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# Supplementary Material: Controlling and Sampling Visibility Information on the Image Plane

Christian Lessig

Otto-von-Guericke Universität Magdeburg

## 1. Sampling Expansions for Arbitrary Sampling Locations

In the following, we summarize the central ideas required to construct sampling expansions for arbitrary sampling locations $\lambda_k$. We refer to [LDF14] for further details.

Let $\mathcal{H}(X)$ be a Hilbert-space, defined over a set $X$, where point-evaluation $\delta_x$ is a continuous functional. Then, by the Riesz representation theorem, there exists a function $k_x \in \mathcal{H}(X)$, possibly a different one for every $x \in X$, such that

$$\delta_x(f) = \langle k_x, f \rangle = f(x) \tag{1}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathcal{H}$. The function $k_x(y)$ is known as reproducing kernel.

To see the connection between sampling theorems and reproducing kernels, let $\{\lambda_j\}_{j\in\mathcal{N}}$ be a set of locations with $\lambda_j \in X$. Then, under suitable conditions on the distribution of the $\lambda_j$, the induced set $\{k_{\lambda_j}\}_{j\in\mathcal{N}}$ will be a frame, that is an in general redundant basis for $\mathcal{H}$. This implies the existence of dual functions $\tilde{k}_j(y)$ such that any $f \in \mathcal{H}$ can be written as

$$f(x) = \sum_{j\in\mathcal{N}} \langle f(y), \tilde{k}_j(y) \rangle \, k_j(x) = \sum_{j\in\mathcal{N}} \langle f(y), k_j(y) \rangle \, \tilde{k}_j(x) \tag{2}$$

where the second equality follows by duality and we abbreviated $k_j(x) \equiv k_{\lambda_j}(x)$. Using the reproducing property in Eq. 1, we can write the last equation as

$$f(x) = \sum_{j\in\mathcal{N}} f(\lambda_j) \, \tilde{k}_j(x). \tag{3}$$

Eq. 3 is the general form of a sampling expansion or theorem. The classical Shannon theorem is a special case where $\mathcal{H}(X)$ is the space of Fourier-bandlimited functions over $\mathbb{R}^n$, the $\lambda_n$ are formed by a dilated integer grid and $\tilde{k}_n(x) = k_n(x) = \mathrm{sinc}(x-n)$.

One can work numerically with the above construction using the series representation of $k_x(y)$ in an orthonormal basis $\{\phi_i\}$ for $\mathcal{H}$:

$$k_x(y) = k_y(x) = \sum_i \phi_i(x) \, \phi_i(y). \tag{4}$$

Assuming $\{\phi_i\}$ is $M$-dimensional with $M < \infty$, or has been truncated appropriately, and the same is true for $\{\lambda_j\}_{j\in\mathcal{N}}$, i.e $|\mathcal{N}| = N < \infty$, then the basis representation of the set $\{k_{\lambda_j}\}$ in $\{\phi_i\}$ can be written in matrix form

$$K_\phi = \begin{pmatrix} \phi_1(\lambda_1) & \cdots & \phi_M(\lambda_1) \\ \vdots & \ddots & \vdots \\ \phi_1(\lambda_N) & \cdots & \phi_m(\lambda_N) \end{pmatrix} \in \mathbb{R}^{N\times M}. \tag{5}$$

This provides the change of basis matrix from $\{\phi_i\}$ to the reproducing kernel basis formed by the $k_j(x)$. In fact, it is easy to see that multiplying a signal's basis representations $(f_1, \cdots, f_M)$ in $\{\phi_n\}$ with $K_\phi$ yields the pointwise values $f(\lambda_j)$ that form the basis function coefficients in the expansion in Eq. 3. It is also not hard to see that when $\mathcal{H}(X)$ is the space of polynomials in the monomial basis over $\mathbb{R}$ then $K_\phi$ is the classical Vandermonde matrix.

It can be shown that the basis function coefficients of the dual kernel functions $\tilde{k}_j$ with respect to $\{\phi_i\}$ are given by the column of the sampling matrix $S_\phi = K_\phi^{-1}$. Since the construction of $K_\phi$ only requires the evaluation of the basis functions $\phi_i$ at the sampling locations $\lambda_j$ this provides a practical means to construct the sampling theorem in Eq. 3.

## 2. Convolution of $H(x)$ and Raised Cosine

The raised cosine function $\mathrm{Cr}_{B,\beta}$ is defined in the frequency domain as

$$\hat{\mathrm{Cr}}(\xi) = \begin{cases} 1 & |\xi| < B\tilde{\beta}_- \\ \frac{1}{2}\left(1+\cos\left(\frac{\pi}{2B\beta}\left(|\xi| - B(1-\beta)\right)\right)\right) & B\tilde{\beta}_- \leq |\xi| \leq B\tilde{\beta}_+ \\ 0 & \text{otherwise} \end{cases}$$

with $\tilde{\beta}_- = 1-\beta$ and $\tilde{\beta}_+ = 1+\beta$. It has the spatial representation

$$\mathrm{Cr}_{B,\beta}(x) = \begin{cases} \frac{\pi B}{2}\,\mathrm{sinc}\left(\frac{1}{2\beta}\right) & x = \pm\frac{1}{4B\beta} \\ 2B\,\mathrm{sinc}\left(2Bx\right)\frac{\cos(2\pi B\beta x)}{1-(4B\beta x)^2} & \text{otherwise} \end{cases}$$

For $\beta = 1/n$ the convolution of $H(x)$ and the raised cosine function $\mathrm{Cr}B,\beta(x)$ is for $n = 3,5,7,\cdots$ given by

$$\begin{aligned}\mathrm{SCr}(x) = \frac{1}{8\pi^2}\Bigg(2\pi + (-1)^{\frac{1-\beta}{2\beta}}\Bigg(&\mathrm{Si}\left(\frac{\pi}{2}\frac{\beta_-\bar{\beta}_+}{\beta T}\right) + \mathrm{Si}\left(\frac{\pi}{2}\frac{\beta_+\bar{\beta}_+}{\beta T}\right)\\ &+ \mathrm{Si}\left(-\frac{\pi}{2}\frac{\beta_-\bar{\beta}_-}{\beta T}\right) + \mathrm{Si}\left(-\frac{\pi}{2}\frac{\beta_+\bar{\beta}_-}{\beta T}\right)\Bigg)\\ &- 2\mathrm{Si}\left(\frac{\pi\beta_- x}{T}\right) + 2\mathrm{Si}\left(\frac{\pi\beta_+ x}{T}\right)\Bigg)\end{aligned}$$

and for $n = 2,4,\cdots$ by

$$\begin{aligned}\mathrm{SCr}(x) = \frac{1}{8\pi^2}\Bigg(2\pi + (-1)^{\frac{1}{2\beta}}\Bigg(&-\mathrm{Ci}\left(\frac{\pi}{2}\frac{\beta_-\bar{\beta}_+}{\beta T}\right) + \mathrm{Ci}\left(\frac{\pi}{2}\frac{\beta_+\bar{\beta}_+}{\beta T}\right)\\ &+ \mathrm{Ci}\left(-\frac{\pi}{2}\frac{\beta_-\bar{\beta}_-}{\beta T}\right) - \mathrm{Ci}\left(-\frac{\pi}{2}\frac{\beta_+\bar{\beta}_-}{\beta T}\right)\Bigg)\\ &- 2\mathrm{Si}\left(\frac{\pi\beta_- x}{T}\right) + 2\mathrm{Si}\left(\frac{\pi\beta_+ x}{T}\right)\Bigg)\end{aligned}$$

where $\mathrm{Ci}(x)$ is the cosine-integral. Furthermore

$$\begin{aligned}\beta_- &= \beta - 1\\ \beta_+ &= \beta + 1\\ \bar{\beta}_- &= -2\beta x + T\\ \bar{\beta}_+ &= 2\beta x + T.\end{aligned}$$

## References

[LDF14] LESSIG C., DESBRUN M., FIUME E.: A Constructive Theory of Sampling for Image Synthesis Using Reproducing Kernel Bases. *ACM Transactions on Graphics (Proceedings of SIGGRAPH 2014) 33*, 4 (jul 2014), 1–14. 1