

Euler Operators for Stratified Objects with Incomplete Boundaries

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Abstract

Stratified objects such as those found in geometry-based systems (e.g. CAD systems and animation systems) can be stepwise constructed and manipulated through Euler operators. The operators proposed in this paper extend prior operators (e.g. the Euler-Masuda operators) provided that they can process n -dimensional stratified subanalytic objects with incomplete boundaries. The subanalytic objects form the biggest closed family of geometric objects defined by analytic functions. Basically, such operators are attachment, detachment, subdivision, and coalescence operations without a prescribed order, providing the user with significant freedom in the design and programming of geometric applications.

Categories and Subject Descriptors (according to ACM CCS): I.3.5 [Computational Geometry and Object Modeling]: Curve, surface, solid, and object representations.

1. Introduction

In geometric modeling and design it is frequently desirable to permit operations which violate the compactness condition inherent to most academic and commercial geometric kernels. For example, subtracting two surface-overlapping solid objects originates a solid object with part of its surface missing, i.e. a non-compact or boundary-incomplete solid object. To overcome this problem, most geometric kernels use a kind of regularization operator to maintain their geometric consistency. Also, a small geometric change of an object might be carried out quickly if we were allowed to break down the boundary completeness condition. Instead, that requires its complete re-design and reconstruction. Besides, many drafting and design activities are geometrically boundary-incomplete (or non-compact in the usual relative topology). For example, we use revolution axes and symmetry lines in modeling, which are not closed nor bounded. Also, the design of an artifact sometimes starts from an erratic set of lines, not from a raw block as in sculpture. This discussion suggests that practical deficiencies of existing geometric kernels are in general due to theoretical restrictions imposed by their supporting mathematical models. While the pioneering geometric kernels were overconstrained by the notion of solidity, the current geometric kernels are constrained to keep the boundary-completeness of geometric objects.

2. Related work

Conceptually, a geometric kernel can be given a triangular architecture: geometry (shape), structure, and algebra. Geometry determines the geometric coverage of the objects. A structure has to do with the topological coverage of these objects, as well as their constituents. An algebra basically concerns the essential operators used to build up and manipulate geometric objects.

2.1. Subanalytic geometry

Recently, the subanalytic geometry has been proposed as an appropriate family of objects for geometric modeling [GMR99], [MRG99], [MRG00]. By abuse of language, we say that the subanalytic geometry consists of subanalytic point sets. A subanalytic set $X \subseteq \mathbb{R}^m$ is defined by the intersecting set of a family $\{x \in \mathbb{R}^m : f(x) \geq 0\}$ of sets described by analytic equalities (zero sets) and inequalities (positive or negative sets), where $f : \mathbb{R}^m \rightarrow \mathbb{R}$ is an analytic function. Real analytic functions include polynomial, rational, and transcendental functions. Briefly speaking, subanalytic sets are important by the following reasons:

- They provide a wide geometric coverage in \mathbb{R}^n , including the geometries commonly used in solid modeling and free-form modeling of curves and surfaces [Gom00], i.e. the semialgebraic sets of the first CSG and B-Rep modelers, as well as the semianalytic sets that underpin CNRG [RR91] and SGC [RO90] representations. Although Bézier curves and surfaces—as well NURBS (Non-Uniform Rational B-Splines)—are not zero sets, they are also semialgebraic because Tarski and Seidenberg proved that semialgebraicity is preserved by rational maps, i.e. if $X \subseteq \mathbb{R}^m$ is semialgebraic, and $f : X \rightarrow \mathbb{R}^n$ is rational, then the image set $f(X)$ is also semialgebraic in \mathbb{R}^n [Tar51] [Sei54].
- They form a Boolean class, i.e. the set-combination of any two subanalytic sets is always a subanalytic set [Hir73].
- Each topological operation such as interior (Int), frontier (Front), exterior (Ext), closure (Clo) on a subanalytic set X has a subanalytic set as its result [Hir74]. In fact, assuming that the interior $\text{Int}(X)$ is well-defined, it follows from the Boolean class that the boundary $\text{Bd}(X) = X \setminus \text{Int}(X)$, the frontier $\text{Fr}(X) = \text{Bd}(X) \cup \text{Bd}(\mathbb{R}^m \setminus X)$, and closure $\text{Cl}(X) = \text{Int}(X) \cup \text{Fr}(X)$ of a subanalytic set X in \mathbb{R}^m is also a subanalytic set. For example, the open disc $X = \mathbb{D}^2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ coincides with its interior;

hence, $\text{Bd}(\mathbb{D}^2) = \emptyset$, $\text{Fr}(\mathbb{D}^2) = \mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, and $\text{Cl}(\mathbb{D}^2) = \mathbb{D}^2 \cup \text{Fr}(\mathbb{D}^2)$. Note that, in general, $\text{Bd}(X) = \text{Fr}(X)$ only for closed sets.

- They admit regular stratifications. A stratification Σ of a point set X is a partition of X into subsidiary point sets called strata, which are by definition submanifolds (e.g. cells). A *stratified* point set $\mathbf{X} = (X, \Sigma)$ distinguishes from another by imposing different regularity conditions on the strata. A regularity condition determines how the strata fit together, i.e. their adjacency and incidence relations. A *subanalytic* Verdier stratification is a stratification in which all strata are subanalytic submanifolds of \mathbb{R}^n and satisfy the Verdier condition [Ver76]. But, there are many other ways to impose a regularity condition on incident and adjacent strata (see [Sha91] [Gom00] [MRG00] for some criteria).

2.2. Stratified objects

Both CSGs and CNRGs represent a point set X by means of a covering of point sets in some Boolean class, while B-Reps and SGCs represent stratified sets. Recall that CSG and CNRG boundary evaluators can be used to output B-Rep and SGC objects, respectively. A theorem due to Verdier formalizes the relation between the *geometry* (e.g. the subanalytic geometry) and the *topology* (e.g. stratum complex or stratification) found in the B-Rep data structures, by stating that every subanalytic set is Verdier-stratifiable [TW99]. Thus, Verdier-stratified subanalytic objects constitute an adequate mathematical model for B-Reps [Gom03].

Stratified sets generalize cell complexes in that an n -cell is a particular n -stratum that is homeomorphic to \mathbb{R}^n . Unlike a cell, a stratum need not be connected, nor bounded, nor globally homeomorphic to an open ball. For example, the 1-sphere $\mathbb{S}^1 = \{p \in \mathbb{R}^2 : \|p\| = 1\}$ admits a stratification of a single 1-stratum, while a cell complex requires at least two cells, e.g. a 1-cell and a 0-cell. Although most B-Rep data structures represent stratified sets, not just cell complexes, they lack generality even in \mathbb{R}^3 . Usually, the construction of a solid object starts with a topological surface homeomorphic to $\mathbb{S}^2 = \{p \in \mathbb{R}^3 : \|p\| = 1\}$ stratified into a 2-cell and a 0-cell by calling the Euler operator *mvfS* (make vertex, face, and shell). This is so because the mathematical model ruled by the Euler formula does not include strata homeomorphic to spheres; hence, the inclusion of a vertex. In contrast, the Euler formula proposed here admits strata homeomorphic to spheres or even tori.

3. Euler algebra

By definition, an algebra is a set of elements together with a set of operations. In this paper our elements are stratified subanalytic sets and the operations are Euler operators. Euler operators are shape operators, i.e. they change the shape of an object in conformity with some Euler formula. These objects and their strata may have incomplete boundaries. Thus, the Euler algebra proposed here generalizes the Euler algebra introduced by Masuda *et al.* [MSNK89], which in turn generalises other formulas (e.g. [Bau72], [Wei86], [Wu89], [YK95]) underpinning manifold and non-manifold boundary representations of geometric objects.

3.1. Euler formula

Understanding the shape of point sets, either stratified or not, is essential in the design of B-Rep data structures and their Euler op-

erators [Gom03]. Basically, these operators process topological and homotopic shapes of both strata and stratified objects by using particular shape invariants. For example, the dimension of a stratum is a topological invariant; consequently, a vertex and an edge possess distinct topological shapes provided that they have distinct dimensions. On the other hand, an n -stratum may have a number of k -holes ($0 \leq k \leq n$). The number of k -holes of a stratum is a homotopic invariant. Two strata with distinct numbers of k -holes are said to have distinct homotopic shapes. The number of k -holes is also known as the k th Betti number. The arrangement of strata and their holes in a stratified object is characterized as follows:

Theorem 1 Let $\mathbf{X} = (X, \Sigma)$ be a regular stratified subanalytic set in \mathbb{R}^3 , where X is its underlying point set and Σ is its Verdier stratification (or set of strata). The Euler characteristic of Σ is

$$\chi(\Sigma) = v - (e - e_h) + (f - f_h + f_c) - (s - s_h + s_c) + \chi_\infty(\Sigma) \quad (1)$$

with $\chi_\infty(\Sigma) = -(e_\infty) + (f_\infty - f_{\infty h}) - (s_\infty - s_{\infty h} + s_{\infty c})$, and where v , e , f , and s stand for the number of boundary-complete vertices, edges, faces, and solids in X , respectively; e_∞ , f_∞ , and s_∞ denote the number of boundary-incomplete edges, faces, and solids in X ; e_h , f_h , and s_h stand for the number of boundary-complete 1-holes through edges, faces, and solids, respectively; $f_{\infty h}$ and $s_{\infty h}$ stand for the number of boundary-incomplete 1-holes through faces and solids, respectively; f_c and s_c indicate the number of boundary-complete 2-holes for faces and solids, respectively, while $s_{\infty c}$ indicates the boundary-incomplete 2-holes in solids, respectively. ■

According to (1), a *vertex* has no holes. An *edge* admits 1-holes (e_h). For example, a ring is an edge with a 1-hole. An edge with a single 1-hole ($e_h = 1$) has the homotopy type of a 1-sphere \mathbb{S}^1 , i.e. it can be continuously deformed to a circle. A *face* may present several 1-holes (i.e. through holes) and a 2-hole (i.e. a void). In the latter case, a face is homotopy-equivalent to the 2-sphere \mathbb{S}^2 . In the formula (1), f_h and f_c stand for 1-holes and 2-holes of faces, respectively. Another example of a face with a 2-hole is the toroidal surface \mathbb{T}^2 . It also has two 1-holes because we can draw two imaginary loops on it, which are not contractible to a point, nor contractible to each other. It is then said that \mathbb{T}^2 has the homotopy type of $\mathbb{S}^1 \times \mathbb{S}^1$, i.e. two loops or rings intersecting at a single point. This means that \mathbb{T}^2 can be formed by sweeping the first ring \mathbb{S}^1 along the second touching ring \mathbb{S}^1 . Filling in \mathbb{T}^2 with an open solid torus (3-manifold) one obtains a closed solid torus (closed 3-manifold). This filling operation makes the 2-hole of \mathbb{T}^2 disappear along with one of its 1-holes (the sweeping ring \mathbb{S}^1). That is, only one 1-hole (the revolution ring of \mathbb{T}^2) remains in the closed solid torus.

It is assumed that, as a topological space, an n -dimensional manifold (or, simply, an n -manifold) is a point set topologized by the usual topology in \mathbb{R}^n . Thus, every n -manifold is open in \mathbb{R}^n , not necessarily bounded, with possibly many k -dimensional holes ($0 \leq k < n$). Unbounded, or equivalently boundary-incomplete, strata appear with the subscript ∞ . For example, in Figure 17(d), with the exception of the vertex v_1 all strata are unbounded. Besides, as suggested by (1), unbounded strata may also have holes. The Euler characteristic (1) regulates the manifold strata of a stratified object, as well as their homotopic shapes (k -dimensional holes) via Betti numbers. Thus, formula (1) provides us with a shape understanding at the stratum level. To describe the shape of a stratified object as a whole, we use the global Euler characteristic as follows:

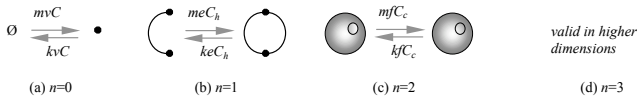


Figure 1: The operator $ms^n H^n$.

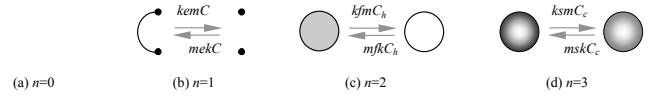


Figure 3: The operator $ks^n mH^{n-1}$.

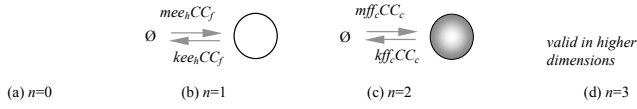


Figure 2: The operator $ms^n h_n^n H^0 H^n$.

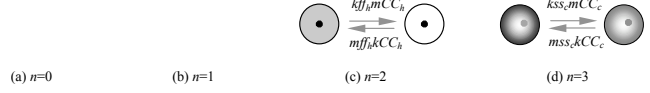


Figure 4: The operator $ks^n h_n^{n-1} mH^0 H^{n-1}$.

Theorem 2 Let $\mathbf{X} = (X, \Sigma)$ be a regular stratified subanalytic set in \mathbb{R}^3 . The Euler characteristic of X is

$$\chi(X) = (C - C_h + C_c) + \chi_\infty(X) \quad (2)$$

with $\chi_\infty(X) = -(E_\infty) + (F_\infty - F_{\infty h}) - (S_\infty - S_{\infty h} + S_{\infty c})$, and where $C, C_h,$ and C_c stand for the number of boundary-complete components, 1-holes, and 2-holes of X , respectively; $E_\infty, F_\infty,$ and S_∞ denote the number of boundary-incomplete components for edges, faces, and solids, respectively; $F_{\infty h}$ and $S_{\infty h}$ denote the number of boundary-incomplete 1-holes through face components and solid components, respectively, and $S_{\infty c}$ the number of boundary-incomplete 2-holes in solid components. ■

Note that the global or homotopic shape of the whole space X underlying \mathbf{X} can be described by the number of 0-holes (or object components), 1-holes, ..., k -holes, which are denoted by $H^0, H^1, \dots, H^k, 0 \leq k \leq n$, respectively. The n -dimensional counterparts of (1) and (2) can be found in [Gom00].

3.2. Euler operators

Recall that a stratum needs not be boundary-complete. Therefore, there is no prescribed order in attaching or detaching a stratum to or from an object. That is, unlike the conventional boundary representations, it is no longer necessary to follow the precedence principle that attaching a stratum must be done after attaching its frontier strata. This makes it possible to proceed to local shape changes on an object without rebuilding the whole object up.

Global hole shapers. There are two classes of operators to generate a global hole through or in X . The first uses a *stratum attachment* technique, while the second uses a *stratum detachment* technique.

Let us now consider the first class of Euler operators that create a global hole by attaching a stratum:

- (i) $ms^n H^n$. This operator creates a global n -hole H^n in X by attaching an n -stratum s^n to Σ (Figure 1). For example, $meC_h = ms^1 H^1$ is a particular case for dimension $n = 1$, which creates a 1-hole through an object by attaching a 1-stratum to it. The attaching n -stratum must be homeomorphic to \mathbb{R}^n . Its inverse $ks^n H^n$ undoes an n -hole by detaching an n -stratum.
- (ii) $ms^n h_n^n H^0 H^n$. This operator is similar to the previous one, but the attaching n -stratum s^n must possess an n -hole h_n^n , i.e. it is not homeomorphic to \mathbb{R}^n ; the subscript of the n -dimensional hole h_n^n denotes the dimension of its ambient stratum s^n (Figure 2). For example, the operator $mf f_c CC_c = ms^2 h_2^2 H^0 H^2$ adds a 2-sphere

to an object, which locally is a face f with a 2-hole f_c , and globally a surface component C with a 2-hole C_c .

Let us now consider the Euler operators that create a global hole by detaching a stratum:

- (i) $ks^n mH^{n-1}$. This operator generates a global $(n - 1)$ -hole H^{n-1} in X by detaching an n -stratum s^n from Σ (Figure 3). For example, $kfmC_h = ks^2 mH^1$ creates a 1-hole through an object by detaching a 2-stratum. The detaching n -stratum must be homeomorphic to \mathbb{R}^n . Its inverse $ms^n kH^{n-1}$ eliminates a global $(n - 1)$ -hole by attaching an n -stratum.
- (ii) $ks^n h_n^{n-1} mH^0 H^{n-1}$. This operator is similar to the previous one, but the detaching n -stratum s^n possesses a $(n - 1)$ -hole h_n^{n-1} , i.e. it cannot be homeomorphic to \mathbb{R}^n (Figure 4). For example, the operator $kff_h m CC_h = ks^2 h_2^1 mH^0 H^1$ removes a face f with a 1-hole f_h from an object, globally originating the appearance of a new component C and a new 1-hole C_h through the object.

Stratum subdividers. Unlike the previous operators, no stratum subdivider changes the global shape of an object. A subdivider is an Euler operator that subdivides an n -stratum into two n -strata by a new $(n - 1)$ -stratum, called the subdividing stratum. There are three generic stratum subdividers/coalescers:

- (i) $ms^{n-1} s^n$. This operator subdivides an n -stratum into two by a new $(n - 1)$ -stratum (Figure 5). The subdividing $(n - 1)$ -stratum must be homeomorphic to \mathbb{R}^{n-1} . This operator applies to both boundary-complete and boundary-incomplete strata. For example, the operator mve in Figure 5 subdivides an edge independently of whether its boundary is complete or not. Its inverse operator $ks^{n-1} s^n$ coalesces two n -strata into one by merging one of their adjacent $(n - 1)$ -strata to which they are incident.
- (ii) $ms^{n-1} h_{n-1}^{n-1} s^n h_n^{n-1}$. In this case, the subdividing $(n - 1)$ -stratum has an $(n - 1)$ -hole, which means that it is not homeomorphic to \mathbb{R}^{n-1} (Figure 6). For example, the operator $mee_h f f_h = ms^1 h_1^1 s^2 h_2^1$ subdivides a face f into two faces by an edge e with a 1-hole e_h , leaving the original face with a 1-hole f_h .

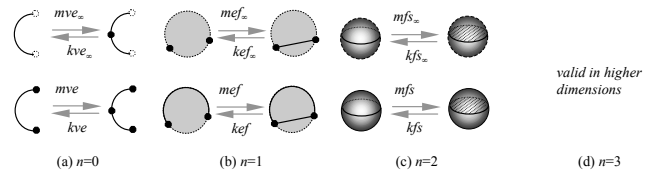


Figure 5: The operator $ms^{n-1} s^n$.

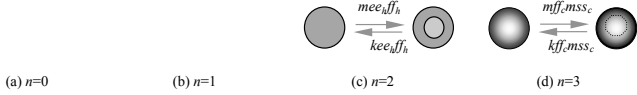


Figure 6: The operator $ms^{n-1}h_{n-1}^{n-1}s^n h_n^{n-1}$.

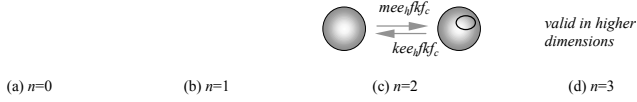


Figure 7: The operator $ms^{n-1}h_{n-1}^{n-1}s^n kh_n^n$.

(iii) $ms^{n-1}h_{n-1}^{n-1}s^n kh_n^n$. In this case, both strata have holes. This operator subdivides a n -stratum with an n -hole by an $(n-1)$ -stratum with an $(n-1)$ -hole (Figure 7). For example, $mee_h f k f_c = ms^1 h_1^1 s^2 k h_2^2$ subdivides a face f with a 2-hole f_c by an edge e with a 1-hole e_h ; this is equivalent to subdividing S^2 by S^1 .

Local hole shapers. They do not change the global shape of an object. They change the shape of strata by merging a stratum into another of higher dimension. The stratum of lower dimension has no holes. There are two sorts of local shapers:

- (i) $ks^d h_n^{n-1-d}$. This merges a d -stratum into an incident n -stratum ($n > d$), which causes the disappearance of a $(n-1-d)$ -hole from the ambient higher dimensional stratum (Figure 8). For example, the operator $kvf_h = ks^0 h_2^1$ eliminates a 1-hole f_h from a face by merging a vertex v (which fills in f_h) into it. Thus, the n -stratum loses an $(n-1-d)$ -hole. This operator works independently of whether the n -stratum is boundary-complete or not.
- (ii) $ks^d mh_n^{n-d}$. In this case, the n -stratum acquires a $(n-d)$ -hole (Figure 9). For example, $kvmf_c$ merges a vertex v into a face, which then closes onto itself, i.e. it acquires a new 2-hole f_c . This operator also works for boundary-incomplete strata. For example, in Figure 9(h), $ke_\infty ms_c$ merges an edge with a vertex missing into a solid having a vertex filling in one of its holes s_c .

Stratum attachers. These Euler operators allow us to attach (respectively, detach) boundary-incomplete strata to (respectively, from) an object. The attaching (detaching) n -stratum must be homeomorphic to \mathbb{R}^n . There is only one class of these operators:

- (i) $ms_\infty^n S_\infty^n$. This operator attaches a boundary-incomplete n -stratum s_∞^n and its corresponding boundary-incomplete stratum

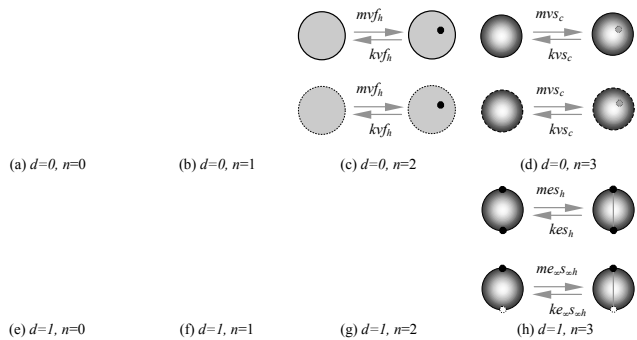


Figure 8: The operator $ks^d h_n^{n-1-d}$.

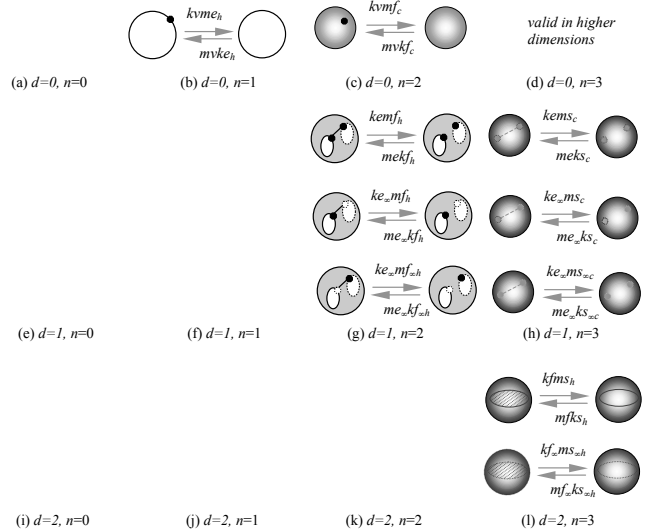


Figure 9: The operator $ks^d mh_n^{n-d}$.

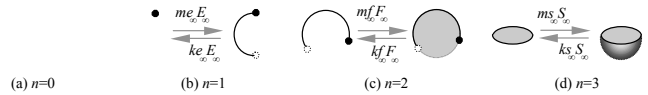


Figure 10: The operator $ms_\infty^n S_\infty^n$.

component S_∞^n to an object (Figure 10). Its inverse $ks_\infty^n S_\infty^n$ detaches a boundary-incomplete n -stratum from an object.

Unfilled hole shapers. They generate boundary-incomplete holes for strata. There are three unfilled hole shapers in \mathbb{R}^3 (Figure 11). They are in the following dimension-independent class:

- (i) $mh_{\infty n}^d H_{\infty n}^d$. It makes a d -hole $h_{\infty n}^d$ ($d \leq n-1$) in a boundary-incomplete n -stratum, which in turn produces a global d -hole $H_{\infty n}^d$ in its stratum component. For example, the operator $mf_{\infty h} F_{\infty h} = mh_{\infty 2}^1 H_{\infty 2}^1$ makes a 1-hole $f_{\infty h}$ through a face and, consequently, a 1-hole $F_{\infty h}$ through its face component.

Local stratum compacters. There is only one dimension-independent stratum compacter:

- (i) $ks_\infty^n ms_\infty^n$. This operator transforms a boundary-incomplete n -stratum s_∞^n into a boundary-complete stratum s^n (Figure 12). For that, we have only to change its boundary-completeness state. For example, in Figure 12(b), attaching a vertex to a boundary-

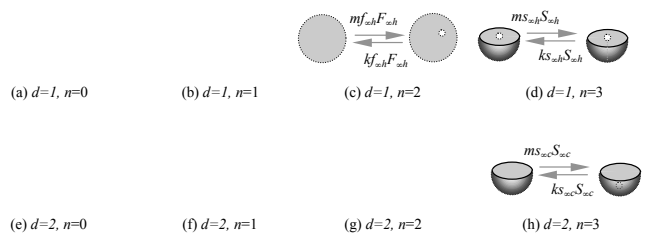


Figure 11: The operator $mh_{\infty n}^d H_{\infty n}^d$.

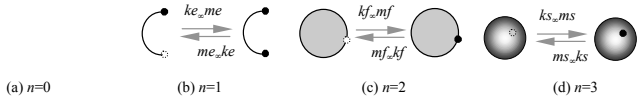


Figure 12: The operator $ks_{\infty}^n ms^n$.

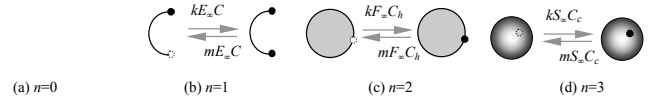


Figure 14: The operator $kC_{\infty}^n H^{n-1}$.

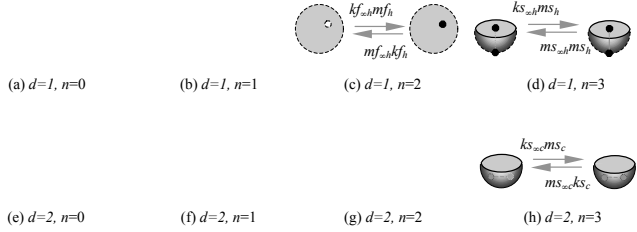


Figure 13: The operator $kh_{\infty}^d mh_n^d$.

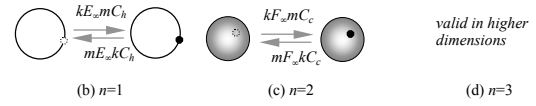


Figure 15: The operator $kC_{\infty}^n mH^n$.

incomplete edge is done by calling $ke_{\infty}me = ks_{\infty}^1ms^1$ to change the boundary-completeness state of the original edge.

A stratum compacter is invoked whenever attaching or detaching a stratum changes the boundary-completeness of its neighboring strata. Note that the compactification of a stratum requires various Euler operators. For example, compacting the face in Figure 12(c) starts by calling the operator mvC , and then the stratum compacters for such a face and its boundary-incomplete edge.

Local hole compacters. Local holes can also be compacted. This is done by filling it in with an appropriate stratum as follows:

(i) $kh_{\infty}^d mh_n^d$. It transforms a boundary-incomplete hole h_{∞}^d of a n -stratum into a boundary-complete hole after filling it in with a stratum of dimension less than n . For example, $kf_{\infty}^1 mf_h^1 = kh_{\infty}^1 mh_2^1$ transforms f_{∞}^1 into a boundary-complete face hole f_h after filling it with a vertex. (This vertex is attached by calling the operator mvC .)

Global stratum compacters. The local compactification of a stratum s_{∞}^n leads to the global compactification of its associated component S_{∞}^n . Thus, a global compacter of a stratum serves to compact the global shape of a stratum. There are three families of global compacters in \mathbb{R}^3 :

- (i) $kC_{\infty}^n H^{n-1}$. It compacts an n -stratum component C_{∞}^n by deleting a global $(n-1)$ -hole H^{n-1} . Consequently, such a component becomes part of the boundary-complete subset of the object, decreasing so the number of boundary-incomplete components. For example, the operator $kE_{\infty}C = kC_{\infty}^1 H^0$ in Figure 14(b) compacts an edge component E_{∞} by deleting the component C of the compacting vertex attached to the object in the meanwhile.
- (ii) $kC_{\infty}^n mH^n$. It generates a global n -hole H^n by compacting an n -stratum component C_{∞}^n . For example, $kE_{\infty}mC_h = kC_{\infty}^1 mH^1$ in Figure 15(b) generates a global hole C_h through the object by attaching a compacting vertex to an edge component E_{∞} .
- (iii) $kh_{\infty}^{n-1}C$. This operator applies to boundary-incomplete stratum components with holes. It compacts a global non-compact $(n-1)$ -hole H_{∞}^{n-1} of a n -stratum after filling it in with a stratum of dimension less than n (Figure 16). Consequently, the filling stratum form a component C which is then merged with some component of the boundary-complete subset of the object, decreasing so the number of components of the whole object.

4. Examples

Let us now illustrate the construction of some stratified objects:

Example 1. The stratified Cartan umbrella, whose underlying point set $x^2 = zy^2$ is semialgebraic (Figure 17) can be built up through the following Euler operators:

- mvC . It creates the vertex v_1 at the origin $(0,0,0)$. This vertex embodies a single component C of the object.
- $me_{\infty}E_{\infty}$. It is used twice to create two unbounded edges e_1 and e_2 (i.e. the negative and positive z -axes), and their corresponding boundary-incomplete edge components E_{∞} , attached to v_1 .
- $mf_{\infty}F_{\infty}$. It is applied twice to create the two unbounded sheets f_1 and f_2 , as well as their corresponding boundary-incomplete face components F_{∞} , attached to e_1 and e_2 , respectively.

Example 2. Let us now construct the compact non-manifold stratified object pictured in Figure 18:

- mee_hCC_h . It creates the edge e_1 with a 1-hole e_h through it. The result is an object with a component C having a 1-hole C_h .
- $mfkC_h$. The global hole C_h produced before disappears by filling it in with the face f_1 .
- mfC_c . Attaching the new face f_2 to e_1 produces a void C_c .
- mvf_h . Attaching the new vertex v_1 to f_1 produces a new 1-hole f_h through f_1 . The global shape of the object remains unchanged.
- meC_h . Finally, attaching the new edge e_2 to v_1 produces a new global hole C_h through the final object.

Note that these objects cannot be constructed by using other Euler operators found in the literature because they cannot handle unbounded strata (e.g. edges and faces of the Cartan umbrella in Figure 17) and edges with holes (e.g. the edge e_1 in Figure 18).

5. Conclusions

The Euler operators that have been proposed in this paper are dimension-independent and can cope with boundary-incomplete stratified objects. This has enabled the construction of objects without a pre-defined order in attaching and detaching strata to and from

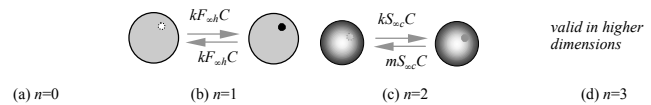


Figure 16: The operator $kh_{\infty}^{n-1}C$.

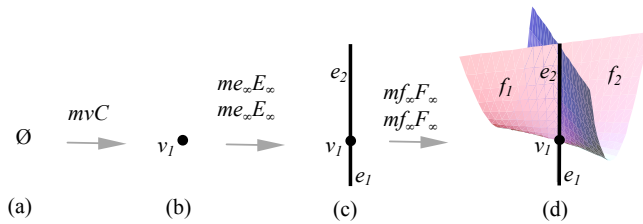


Figure 17: Stratified Cartan umbrella.

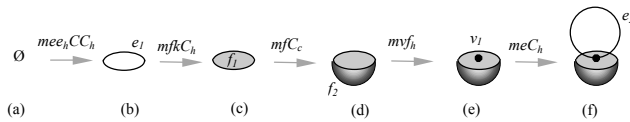


Figure 18: A compact non-manifold stratified object.

them, respectively. Thus, the precedence principle of conventional boundary representations no longer needs to be satisfied. This is useful for many geometry-based applications where significant freedom is required in the design of geometric artifacts. Finally, the fact that these Euler are dimension-independent facilitates the implementation and maintenance of the geometric kernel.

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