Eurographics Symposium on Geometry Processing (2017)
J. A. Bærentzen and K. Hildebrandt (Editors)

Localized Manifold Harmonics for Spectral Shape Analysis

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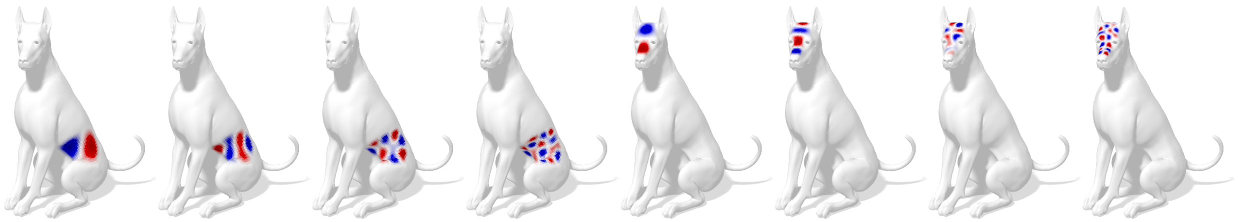

Figure 1: *A few Localized Manifold Harmonics (LMH) on two different regions. By changing the region location on the surface, our model provides an ordered set of* localized *harmonic functions (i.e., defined on the entire surface, but strongly concentrated on the selected region). In this figure the localized harmonics are clearly visible across different frequencies. The LMH constitute a valid alternative to the classical manifold harmonics and can be used in conjunction with those, or as a drop-in replacement in typical spectral shape analysis tasks.*

Abstract

The use of Laplacian eigenfunctions is ubiquitous in a wide range of computer graphics and geometry processing applications. In particular, Laplacian eigenbases allow generalizing the classical Fourier analysis to manifolds. A key drawback of such bases is their inherently global nature, as the Laplacian eigenfunctions carry geometric and topological structure of the entire manifold. In this paper, we introduce a new framework for local spectral shape analysis. We show how to efficiently construct localized orthogonal bases by solving an optimization problem that in turn can be posed as the eigendecomposition of a new operator obtained by a modification of the standard Laplacian. We study the theoretical and computational aspects of the proposed framework and showcase our new construction on the classical problems of shape approximation and correspondence.

Categories and Subject Descriptors (according to ACM CCS): I.3.5 [Computer Graphics]: Computational Geometry and Object Modeling—Shape Analysis, 3D Shape Matching, Geometric Modeling

CCS Concepts
•***Computing methodologies*** → ***Shape analysis;***

1. Contribution

One of the key disadvantages of the standard Laplacian eigenbases used in spectral geometry processing applications [VL08] is their *global* support: representing local structures may require using (potentially, infinitely) many basis functions. In many applications, one wishes to have a local basis that allows to limit the analysis to specific parts of the shape. The recently proposed *compressed manifold harmonics* [OLCO13] attempt to construct local orthogonal bases that approximately diagonalize the Laplacian, but do not allow to explicitly control the localization of the basis functions.

We propose a new type of intrinsic operators whose spectral decomposition provides a local basis. The overall aim is to integrate the *global* information obtained by the Laplacian eigenfunctions with *local* details given by our new basis. The *Localized Manifold Harmonic (LMH)* basis constructed this way is

- smooth, local, and orthogonal. In particular, it is possible to construct the localized basis in an incremental way, such that the new functions are orthogonal to some given set of functions (e.g., standard Laplacian eigenfunctions);
- localized at specified regions of the shape;
- efficiently computed by solving a standard eigendecomposition.

2. Approach

Let us be given a manifold $\mathcal{X}$, a region $R \subseteq \mathcal{X}$ thereof, a set of orthonormal functions $\phi_1,\ldots,\phi_{k'}$ (e.g. the first k' Laplacian eigenfunctions), and an integer k. We seek a new set $\psi_1,\ldots,\psi_k$ of functions that are *smooth*, *orthonormal*, and *localized* on R, as the solu-

DOI: 10.2312/sgp.20171203

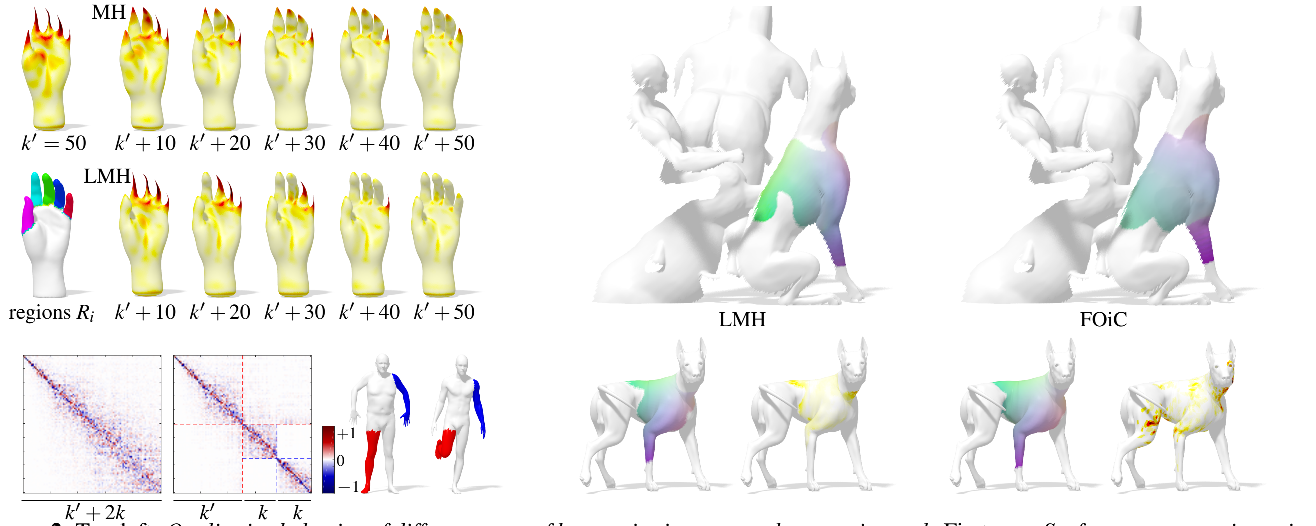

Figure 2: Top-left*: Qualitative behavior of different types of harmonics in a spectral processing task.* First row*: Surface reconstruction using the first k' to $k'+50$ standard manifold harmonics (MH).* Second row*: We use 10 localized manifold harmonics on each region R_i. Note the significantly higher accuracy of LMH despite using the same number of harmonics. The heatmap encodes reconstruction error, growing from white to dark red.* Bottom-left*: Functional map matrices w.r.t. the MH (left) and w.r.t. a "mixed" basis composed of k' MH and $k+k$ LMH (middle). The maps encode the ground-truth correspondence between the two human shapes; the regions used for the computation of LMH are highlighted in red and blue. Note the block-diagonal structure of the second matrix, a manifestation of the capability of LMH to encode local information compactly.* Right*: Qualitative comparison between our LMH-based approach for deformable shape correspondence in clutter and the state of the art [CRM*16] (corresponding points have same color, geodesic error is shown as heatmap).*

tion to the following optimization problem:

$$\min_{\psi_1,\dots,\psi_k} \sum_{j=1}^{k} \mathcal{E}(\psi_j) \quad \text{s.t.} \quad \langle \psi_i, \psi_j \rangle_{L^2(\mathcal{X})} = \delta_{ij} \tag{1}$$

where $\mathcal{E}(\psi_j) = \mathcal{E}_S(\psi_j) + \mu_R \mathcal{E}_R(\psi_j) + \mu_\perp \mathcal{E}_\perp(\psi_j)$. The first term $\mathcal{E}_S$ is the Dirichlet functional promoting the *smoothness* of the new basis. The term

$$\mathcal{E}_R(f) := \int_{\mathcal{X}} (f(x)(1-u(x)))^2 \, \mathrm{d}x\,, \tag{2}$$

is a quadratic penalty promoting the localization of the basis functions on the given region $R \subseteq \mathcal{X}$. Here $u : \mathcal{X} \to [0,1]$ is a membership function such that $u(x) = 1$ for $x \in R$ and $u(x) = 0$ otherwise. Finally, the term

$$\mathcal{E}_\perp(f) := \sum_{i=1}^{k'} |\langle \phi_i, f \rangle_{L^2(\mathcal{X})}|^2 \tag{3}$$

demands the basis functions to be *orthogonal* to the subspace $\text{span}\{\phi_1,\dots,\phi_{k'}\}$. It allows to construct an incremental set of functions that are orthogonal to a given set of Laplacian eigenfunctions.

Let $\mathbf{\Psi} \in \mathbb{R}^{n\times k}$ be a matrix containing our discretized basis functions $\psi_1,\dots,\psi_k$ as its columns, and let $\mathbf{\Phi} \in \mathbb{R}^{n\times k'}$ be a matrix of the first k' Laplacian eigenfunctions $\phi_1,\dots,\phi_{k'}$. Then, the total energy is discretized as $\sum_{j=1}^{k} \mathcal{E}(\psi_j) = \mathcal{E}(\mathbf{\Psi})$, with

$$\mathcal{E}_S(\mathbf{\Psi}) = \text{tr}(\mathbf{\Psi}^\top \mathbf{W} \mathbf{\Psi}) \tag{4}$$

$$\mathcal{E}_R(\mathbf{\Psi}) = \text{tr}(\mathbf{\Psi}^\top \mathbf{A}\, \text{diag}(\mathbf{v}) \mathbf{\Psi}) \tag{5}$$

$$\mathcal{E}_\perp(\mathbf{\Psi}) = \text{tr}(\mathbf{\Psi}^\top \underbrace{\mathbf{A}\mathbf{\Phi}\mathbf{\Phi}^\top \mathbf{A}}_{\mathbf{P}_{k'}} \mathbf{\Psi}) \tag{6}$$

where $\mathbf{v} = ((1-u(x_1))^2, \dots, (1-u(x_n))^2)^\top$.

The discrete version of problem (1) can now be expressed as

$$\min_{\mathbf{\Psi} \in \mathbb{R}^{n\times k}} \text{tr}(\mathbf{\Psi}^\top \mathbf{Q}_{v,k'} \mathbf{\Psi}) \quad \text{s.t.} \quad \mathbf{\Psi}^\top \mathbf{A} \mathbf{\Psi} = \mathbf{I}\,, \tag{7}$$

where the matrix $\mathbf{Q}_{v,k'} = \mathbf{W} + \mu_R \mathbf{A}\text{diag}(\mathbf{v}) + \mu_\perp \mathbf{P}_{k'}$ is symmetric and positive semi-definite. Problem (7) is equivalent to the *generalized eigenvalue problem* $\mathbf{Q}_{v,k'}\mathbf{\Psi} = \mathbf{A}\mathbf{\Psi}\mathbf{\Lambda}$ and can be solved *globally* by classical Arnoldi-like methods.

3. Applications

Localized manifold harmonics are a general tool that can be employed as a drop-in replacement for, or in conjunction with the classical manifold harmonics ubiquitous in spectral shape analysis. We showcase their application in two broad tasks in graphics: spectral shape processing and shape correspondence. We refer to Figure 2 for a discussion of the results.

References

[CRM*16] Cosmo L., Rodolà E., Masci J., Torsello A., Bronstein M. M.: Matching deformable objects in clutter. In *Proc. 3DV* (Oct 2016), pp. 1–10. 2

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