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# Sequentially-Defined Compressed Modes via ADMM

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**Abstract**

*The eigenfunctions of the discrete Laplace–Beltrami operator have played an important role in many aspects of geometry processing. Given the success of sparse representation methods in areas such as compressive sensing it is reasonable to find a sparse analogue of LBO eigenfunctions. This has been done by Ozoliņš et al for Euclidean spaces and Neumann et al for surfaces where the resulting analogues are called compressed modes.*

*In this short report we show that the method of Alternating Direction Method of Multipliers can be used to efficiently calculate compressed modes and that this compares well with a recent method to calculate them with an Iteratively Reweighted Least Squares method.*

## 1. Introduction

Compressed modes were introduced in [OLCO13] and [OLCO14] to provide a basis of the space of functions on Euclidean spaces with the intention of solving PDEs. The elements of this basis differed from the standard eigenfunctions basis in that their support is localised rather than over the whole domain. This can be seen in Figure 1 for compressed modes on surfaces which were introduced in Neumann et al, [NVT*14]. Shaded areas of the six modes indicate that the mode is non-zero.

In [NVT*14] they used the Alternating Direction Method of Multipliers (ADMM) method to calculate a batch of compressed modes all in one go. This required a parameter, $\mu$, which measured the size of the support. The method was accelerated in [Hou15] via a Nesterov method and a method for ordering the modes was also given.

However, problems remained in that if one calculated the first 20 modes say, and then required 10 more, one had to start from the beginning and calculate the first 30. Furthermore, due to an issue with the parameter $\mu$ the first 20 of the new 30 are not the same as the 20 originally calculated.

Another problem was that the definition of compressed modes involved an $\ell_1$ norm term that did not take into account non-uniformity of sampling of the manifold in the discrete case. This was taken care of in [BCKS16]. Furthermore, they give an Iteratively Reweighted Least Squares Method (IRLS) method to calculate the modes sequentially rather than in a batch as in [NVT*14].

In this brief report to accompany an SGP17 conference poster we outline an ADMM method to calculate the modes sequentially. This is compared to the IRLS method.

## 2. Compressed modes

First we define compressed modes for a matrix pair.

**Definition 2.1** Let $W$ be a symmetric positive semi-definite matrix, $A$ a symmetric positive definite matrix and $\mu$ be a non-negative real number. The $k$th **compressed mode** of $L = A^{-1}W$ with respect to the **compression parameter** $\mu$, denoted $\varphi_k$, is defined inductively as

$$\varphi_k = \arg\min_{\varphi} \varphi^T W\varphi + \mu||A\varphi||_1$$

such that $\varphi^T A\varphi = 1$ and $\varphi_i^T A\varphi = 0$ for all $i = 1,\dots,k-1$.

In [BCKS16] $W$ is the cotangent Laplacian and $A$ is the area matrix. In [BH17], where they use the term $\mu||\varphi||_1$ rather that $\mu||A\varphi||_1$, $W$ is the Hessian of discrete deformation energy and $A$ is a mass matrix.

To solve this optimisation problem we use the ADMM method. Details of this method can be found in [BPC*11]. We use a block method and incorporate an acceleration.

The optimization function is split into the sum of three (rather than the more usual two) functions:

$$\arg\min_{\varphi} \varphi^T W\varphi + \mu||A\varphi||_1 + \iota(\varphi)$$

where the indicator function $\iota$ is defined by

$$\iota(\varphi) = \begin{cases} 0, & \text{if } \varphi^T A\varphi = 1 \text{ and } \varphi_i^T A\varphi = 0, \\ \infty, & \text{otherwise,} \end{cases}$$

where $i = 1,\dots,k$.

We can then reformulate the problem as

$$\min_{\varphi,S,E} \iota(\varphi) + E^T WE + \mu||AS||_1 \text{ such that } \varphi = S,\ \varphi = E.$$



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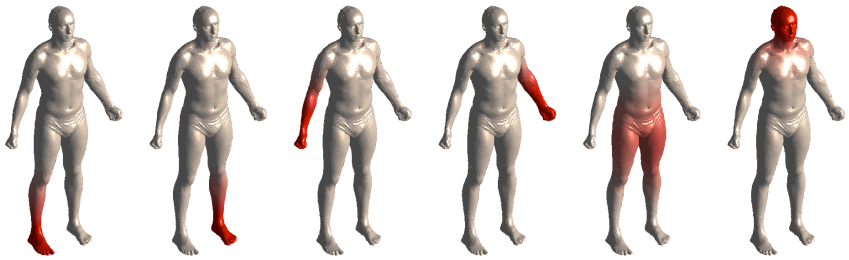

**Figure 1:** *Local support and identification of natural features.*

| | | 10 | 20 | 30 | 40 |
|---|---|---|---|---|---|
| Teapot, 6489 | ADMM | 8.9 | 16.8 | 23.3 | 32.4 |
| vertices, $\mu = 100$ | IRLS | 9.0 | 14.4 | 22.2 | 28.0 |
| Human, 10K | ADMM | 11.4 | 19.7 | 66.9 | 112.5 |
| vertices, $\mu = 625$ | IRLS | 12.8 | 31.9 | 43.4 | 64.1 |
| Aquarius, 100K | ADMM | 102.4 | 312.8 | 557.5 | 846.3 |
| vertices, $\mu = 800$ | IRLS | 42.3 | 180.5 | 396.1 | 630.3 |

**Table 1:** *Time taken (in secs) for the calculation of 10-40 modes in each method for different meshes.*

With this formulation we have to solve three optimization problems and make a simple update. Each of these is fairly straightforward to solve either explicitly or by a simple iterative process.

The IRLS method of [BCKS16] is the following. Suppose we have $k$ compressed modes already. To find $\varphi_{k+1}$ we proceed iteratively. Suppose $\varphi^j$ is an approximation to this compressed mode (the initial $\varphi^0$ could be random). Define a diagonal matrix $\Omega^{j+1}$ with its $p$th diagonal entry equal to $\frac{1}{\varphi^j(p)}$. That is, the $p$th entry on the diagonal is equal to the reciprocal of the $p$th entry in the vector $\varphi^j$. Fortunately, in our case, in practice the entries are non-zero. To avoid division by zero any zero entry can be replaced by a very small value.

We then define the next iterate $\varphi^{j+1}$ by

$$\varphi^{j+1} = \arg\min_{\varphi} \varphi^T \left(W + \mu\Omega^{j+1}A\right)\varphi \qquad (1)$$

such that $\varphi^T A\varphi = 1$ and $\varphi_i^T A\varphi = 0$ for all $i = 1,\ldots,k$.

## 3. Results

The two methods were applied to a number of meshes: The human mesh in Figure 1 (SCAPE dataset, 10K vertices), Aquarius the Water Carrier, (EPFL Computer Graphics and Geometry Laboratory, 100K vertices), and the classic teapot (6489 vertices). The experiments were performed on an iMac with 3.4 GHz Intel core i7 and 8GB RAM using MATLAB/C++.

The times for calculating 10, 20, 30 and 40 modes on each mesh for the two methods is given in Table 1. As it can be seen, the ADMM method is reasonably competitive. However, it was discovered that the IRLS method involves a very poorly conditioned matrix and is not accurately solving an eigenvalue problem that is part of the implementation. Hence, though the method has superior speed it does not necessarily calculate the correct mode.

## 4. Conclusion

The ADMM is very easy to implement compared to the IRLS method and for lower numbers of modes is competitive with respect to speed. However, as the IRLS method involves a poorly conditioned matrix it may not be correctly calculating the modes.

What is needed is a method to analyse the quality of modes produced by the various methods. This and an alternative IRLS method are to be detailed in a paper in preparation.

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