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Poster

Construction of G^3 Conic Spline Interpolation

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Abstract

In this paper, a new method to interpolate a sequence of ordered points with conic splines is presented. The degree of continuity at joints of the resulting splines can reach G^3 while the number of curvature extrema is reduced to a minimum. The construction process is not based on parametrization, but basic geometric elements. A new geometric concept called Chord-Tangent Ratio which is vital to determine the shape of conic splines is proposed. The main idea of the construction is to merge the constraints of continuity into a function of tangent arguments and Chord-Tangent Ratios, and construct an optimization function to eliminate the curvature extrema, then through an iterative process, for the constraint function to reach its zero point and for the optimization function to reach its minimum. Experiments show that splines constructed by the new method performs well not only in terms of continuity, but also in smoothness.

1. Introduction

There are methods to parametrize conics into quadratic rational Bézier curves and join them together with C^1 continuity [Fan02]. It has been proved that conic splines can achieve low interpolation errors when the sample points are taken from a smooth curve [Ann01]. Methods to reduce the error in the sense of Hausdorff distance have also been developed [HwK09]. It has been proved that the conic splines can achieve a higher order approximation with an error less than $O(h^4)$ and a continuity degree of G^2 [Flo95]. A method based on deleting the vertices at joints and on the curves separately, then adjusting the tangents to maintain the the degree of continuity is proposed [Yan04]. A more reliable and robust method to optimize conic splines is required for actual cases. The new method has three advantages: (1) degree of continuity is improved from G^2 to G^3, the highest possible for conic splines. (2) vertices in the splines are eliminated as many as possible, which brings a much fairer shape of splines. (3) the algorithm is simple and robust, so it is easy for application.

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2. Basic Concepts

Given ordered sample points $P_0, P_1, \ldots, P_n$, our goal is to construct a conic spline between every two adjacent samples.

Definition 1 (Chord-Tangent-Ratio) Let t_1 and t_2 be two tangents of conic c, l be the chord through the contact points, P be an arbitrary point on curve c, m be a constant which satisfies

$$m = \frac{d_{P-t_1} \cdot d_{P-t_2}}{d_{P-l}^2},$$

where $d(\cdot,\cdot)$ expresses the distance from a point to a line. We call m the Chord-Tangent Ratio of the section of c.

3. Constraints and Optimization functions

3.1. Constraints

We write the tangent argument at sample point P_i as θ_i, and write the chord length, the chord argument, the Chord-Tangent Ratio and the osculation angles of the i_{th} conic c_i as l_i, m_i, α_i, β_i respectively. The constrain conditions can be expressed by a function:

$$\begin{aligned} & g(\theta,m) \\ =& \sum(\ln m_i + 2\ln\sin(\theta_i-\varphi_i) - \ln\sin(\varphi_i-\theta_{i-1}) - \ln l_i \\ & - \ln m_{i+1} + 2\ln\sin(\varphi_{i+1}-\theta_i) - \ln\sin(\theta_{i+1}-\varphi_{i+1}) \\ & - \ln l_{i+1})^2 + \sum\left(\frac{\sin(\theta_i-\theta_{i-1})}{m_i}\frac{\sin(\varphi_{i+1}-\theta_i)}{\sin(\theta_i-\varphi_{i-1})}\right. \\ & \left. + \frac{\sin(\theta_{i+1}-\theta_i)}{m_{i+1}}\frac{\sin(\theta_i-\varphi_i)}{\sin(\varphi_{i+1}-\theta_i)} - 2\sin(\varphi_{i+1}-\varphi_i)\right)^2. \end{aligned} \tag{1}$$

This function is composed of two summations. The first implies the continuity of curvature; and the second implies the continuity of curvature change rate. This function is an equivalent condition of G^3 continuity about the splines.

3.2. Optimization function

The constraint of continuity cannot finally determine all the elements to construct spline. The main goal to construct an optimization function is reducing the vertices on splines. We reinforce the requirement of spline without vertices into a more rigorous form: The curvature radius change rates at terminals are similar. The optimization function is constructed as:

$$f(\theta,m) = \sum\left(m_i - \frac{1}{2}\left(\frac{\sin(\varphi_i-\theta_{i-1})}{\sin(\theta_i-\varphi_i)} + \frac{\sin(\theta_i-\varphi_i)}{\sin(\varphi_i-\theta_{i-1})}\right)\right)^2. \tag{2}$$

4. Solution of Constrained Optimization Problems

From the discussion above, the construction problem are transformed into a conditional extremum problem. The complete algorithm is shown as following:

1. Estimate the directions of tangents at every points.
2. Calculate the start value of arguments $\{\theta_i\}$ of tangents.
3. Calculate the arguments of the chords $\{\varphi_i\}$ between every two adjacent sample points.
4. Construct the start values of Chord-Tangent Ratios $\{m_i\}$ of every splines.
5. Calculate partial derivatives of the constraint function to $\{\theta_i\}$ and $\{m_i\}$
6. Calculate partial derivatives of the optimization function to $\{\theta_i\}$ and $\{m_i\}$
7. Calculate the descent direction by
$$\mathbf{v} = \left(\nabla g + \left(\nabla f - \frac{(\nabla f\cdot\nabla g)\nabla g}{\nabla g\cdot\nabla g}\right)\right).$$
8. Adjust $\{\theta_i\}$ and $\{m_i\}$ with the descent direction.
9. If more iterations are required, go to step 5.
10. Generate quadratic rational Bézire spline from tangents and Chord-Tangent Ratios.

It is not difficult to find that both the time complexity and space complexity of this algorithm are $O(n)$.

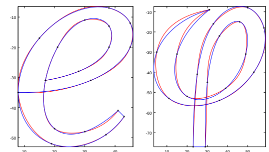

Figure 1: *Letter "e" and "φ"*

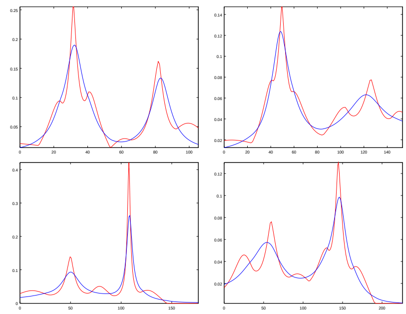

Figure 2: *Curvatures of Splines*

5. Experiments

We depict the latin letter "*e*" and Greek letter "φ" with conics(Figure 1). There are two groups of control points for either of them. One depicts the inside the inner boundary while the other depicts the outer boundary(Figure 1).

From the graphics, we can see conics(blue curve) depict the curves in the letters better than Cubic Hermite Splines(red curve).

The curvature plot(Figure 2) shows us the robustness of conic splines. We can hardly identify the positions where the control point lies. But irregular positions of control points may cause Runge phenomenon on Hermite splines.

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