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Poster

Geodesic Voronoi Diagrams with Polyline Generators

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Abstract

Geodesic Voronoi diagrams (GVDs) defined on triangle meshes with polyline generators are studied in this paper. We introduce a new concept, called local Voronoi diagram, or LVD, which is a weighted Euclidean Voronoi diagram on a mesh triangle. We show that when restricting on a mesh triangle, the GVD is a subset of the LVD, which can be computed by using the existing 2D techniques. Moreover, only two types of mesh faces can contain GVD edges. Guided by our theoretical findings, the geodesic Voronoi diagram with polyline generators can be built in $O(nN\log N)$ *time and takes* $O(nN)$ *space on an n-face mesh with m generators, where* $N = \max\{m,n\}$.

Categories and Subject Descriptors (according to ACM CCS): I.3.5 [Computer Graphics]: Computational Geometry and Object Modeling—Curve, surface, solid, and object representations

1. Introduction

Voronoi diagrams are a fundamental spatial data structure, which is widely used in various engineering fields, such as computational geometry, pattern recognition, robot navigation, wireless network, etc. The Voronoi diagram in Euclidean space have been widely studied and understood. However, Voronoi diagrams defined on triangle meshes based on geodesic metric (also known as geodesic Voronoi diagram or GVD) received little attention. Due to the fundamental difference between Euclidean and geodesic metrics, many Euclidean properties do not hold on meshes.

Inspired by the promising results in [LCT11], in this paper, we investigate the GVD in a more *general* setting, where the generators are polylines. We show that a typical GVD bisector may contain line segments, hyperbolic segments and parabolic segments. Since our situations are more complicated than the 2D Voronoi diagrams as well as the GVD with point sources, we introduce a new concept, called *local Voronoi diagram*, or LVD, which is a weighted Euclidean Voronoi diagram defined on a mesh triangle. We prove that when restricting on a mesh triangle, the GVD is the subset of the LVD. Moreover, only two types of mesh faces can contain GVD edges. Our algorithm can be integrated to the MMP framework in a seamless manner: once the MMP algorithm terminates, both the geodesic distance and the GVD are readily available (Figure 1). We show that GVD can be built in $O(nN\log N)$ time and takes $O(nN)$ space on an n-face mesh with m generators, where $N = \max\{m,n\}$.

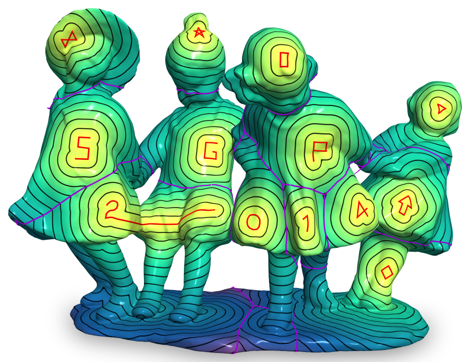


Figure 1: *The geodesic Voronoi diagram (GVD) with polyline generators on triangle meshes. The generators, the bisectors and the iso-distance contours are drawn in red, pink and black, respectively. The background color also indicates the distance to the generators.*

2. GVD with Polyline Generators

Let $M = (V,E,F)$ be the triangle mesh, where V, E and F are the set of vertices, edges and faces, respectively. Given points $p,q \in M$, we denote by $\gamma(p,q)$ the geodesic path between p and q, and $d(p,q)$ the geodesic distance.

Let $\mathcal{G} = \{g_i | g_i \in M, i = 1, \cdots, m\}$ be the set of points. The geodesic Voronoi cell associated with generator g_i is defined as

$$\{x | x \in M, d(x, g_i) \leq d(x, g_j), \forall g_j \in \mathcal{G}\}$$

The bisectors of 2D Voronoi diagram with point generators are always line segments. In our setting, a generator can be either a point or a polyline on the mesh M, which leads to many more complicated situations of bisectors than the Euclidean plane.

Theorem 2.1 Let $g_1, g_2 \in M$ be two distinct generators on the mesh M. Bisector $\beta(g_1, g_2)$ can contain line segments, hyperbolic segments, parabola segments and even a 2D region.

Definition Let w be a window on edge e. The *illuminated region* of w, denoted by $l(w)$, is the region lying on the side which is opposite to $s(w)$. $l(w)$ is bordered by w and the two rays emanating from s.

Definition Given an edge e and two *adjacent* windows $w_1 = (a,b) \in e$ and $w_2 = (b,c) \in e$, their *co-illuminated region*, denoted by $c(w_1, w_2)$, is defined as follows:

1. if $s(w_1)$ and $s(w_2)$ are on the same side of e, $c(w_1, w_2)$ is the intersection of their illuminated regions, i.e., $c(w_1, w_2) = l(w_1) \cap l(w_2)$.
2. otherwise, assume $s(w_2)$ is generator on the other side of e and w_3 is the parent window of w_2. Then $c(w_1, w_2)$ is the intersection of w_1 and w_3's illuminated regions, i.e., $c(w_1, w_2) = l(w_1) \cap l(w_3)$.

Theorem 2.2 Upon the termination of the MMP algorithm, two adjacent windows w_i and w_j have a *non-empty* co-illuminated region. Moreover, bisector $\beta(w_1, w_2)$ is in the co-illuminated region $c(w_1, w_2)$.

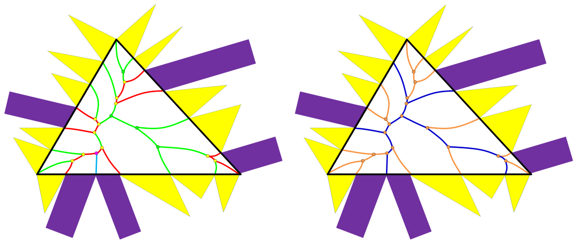
Definition A local Voronoi diagram on a triangle t, denoted by $\mathcal{L}(t)$, is the *weighted* Euclidean Voronoi diagram restricted on t with $\mathcal{P}(t)$ as generators. The weight of a window w is the distance from its pseudo-source to the source if w is a point-source window, and 0 otherwise.

Theorem 2.3 Then each LVD edge bisects two windows, and it does not intersect their borders.

Denote by $\mathcal{G}(t)$ the GVD restricted on triangle t. Obviously, if the mesh has no saddle vertices, the GVD and LVD coincides on t, i.e., $\mathcal{G}(t) = \mathcal{L}(t)$.

Theorem 2.4 The GVD restricted on a triangle t is the subset of the LVD on t, i.e., $\mathcal{G}(t) \subseteq \mathcal{L}(t)$.

Remark Note that the converse of Theorem 2.4 is not true in general. For example, consider two point-source windows $w_1 \in e$ and $w_2 \in e$, which share the *same* generator but have *different* pseudo-sources $s(w_1) \neq s(w_2)$. Obviously, $\beta(w_1, w_2) \in \mathcal{L}(t)$ and $\beta(w_1, w_2) \notin \mathcal{G}(t)$. Figure 2 shows an example of the LVD and the GVD on a triangle.



(a) (b)

Figure 2: *The relationship between GVD and LVD. (a) The LVD edges consist of hyperbolic segments (green), parabolic segments (red) and line segments (cyan). (b) The GVD edges (blue) are the subset of the LVD edges.*

Figure 3: *One GVD example.*

Theorem 2.5 Only two types of triangles, namely, the one with at least one key point on its side, or the one having a source inside, can contain GVD edges.

We implement an algorithm using the above theoretical findings and Figure 3 shows an example.

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References

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