Supplemental Material for:
Hele-Shaw Flow Simulation with Interactive Control using Complex Barycentric Coordinates

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Polubarinova-Galin Equation. We provide the derivation of the PG equation in our notation, to keep the exposition self-contained. It follows directly the derivation in [GV06], pp. 17-18]. Given a complex potential $W_U(\zeta)$ which solves Equations (2a) and (2b) in the unit disk, the solution for the complex potential in $\Omega(t)$ can be obtained by the composition $W_{\Omega(t)}(f(\zeta,t))(z) = W_U(\zeta)$, and thus for its derivative we have:

$$\frac{\partial W}{\partial z}\frac{\partial f}{\partial \zeta} = \frac{\partial W_U}{\partial \zeta}. \tag{1}$$

The unit normal to the boundary of U at a point ζ is simply ζ. Thus, the unit normal $\hat{n}(z)$ to $\Omega(t)$ at a point $z = f(\zeta)$ can be expressed using f as $\zeta f_\zeta / |f_\zeta|$, where $f_\zeta = \partial f / \partial \zeta$. Hence, the normal velocity can be written using the complex potential by:

$$v_n = -\frac{1}{|f_\zeta|}\operatorname{Re}\left(\frac{\partial W_{\Omega(t)}}{\partial z}\zeta f_\zeta\right),$$

and using the conformal map by $v_n = \operatorname{Re}(\frac{\partial f}{\partial t}\overline{\hat{n}})$, or:

$$v_n = \frac{1}{|f_\zeta|}\operatorname{Re}\left(\frac{\partial f}{\partial t}\overline{\zeta f_\zeta}\right).$$

Combining the last two expressions we get:

$$\operatorname{Re}\left(\frac{\partial f}{\partial t}\overline{\zeta\frac{\partial f}{\partial \zeta}}\right) = -\operatorname{Re}\left(\frac{\partial W_{\Omega(t)}}{\partial z}\frac{\partial f}{\partial \zeta}\zeta\right).$$

Plugging in the conformal invariance (1) we have:

$$\operatorname{Re}\left(\frac{\partial f}{\partial t}\overline{\zeta\frac{\partial f}{\partial \zeta}}\right) = -\operatorname{Re}\left(\frac{\partial W_U}{\partial \zeta}\zeta\right). \tag{2}$$

In our case the complex potential which solves Equation (2a) on the unit disk is given by $W_U(\zeta) = \frac{Q}{2\pi}\log\zeta$, as its real part is the Green's function of the Laplacian. Furthermore, it fulfills the boundary conditions (2b), as $\operatorname{Re}(\log\zeta) = 0$ for $|\zeta| = 1$. Thus we have $\partial W_U / \partial \zeta = Q / 2\pi\zeta$, which we plug into Equation (2) and get the PG equation:

$$\operatorname{Re}\left(\frac{\partial f}{\partial t}\overline{\zeta\frac{\partial f}{\partial \zeta}}\right) = -\frac{Q}{2\pi}, \quad \zeta \in \partial U.$$

Line potential. The potential of a line singularity is calculated by integrating the solution of a point singularity at location s (which is given by the Green's function $W_s = \frac{Q}{2\pi}\log(z-s)$) over the line $l : s(t) = z_1 + t(z_2 - z_1)$:

$$\begin{aligned}
W_l &= \int_l \frac{Q}{2\pi}\log(z-s(t))dt \\
&= \frac{Q}{2\pi}\int_0^1 \log(z-(z_1+t(z_2-z_1)))dt \\
&= \frac{Q}{2\pi}\left(\frac{z-s(t)}{z_2-z_1}(1-\log(z-s(t)))\right)\Bigg|_0^1 \\
&= \frac{Q}{2\pi(z_2-z_1)}\Big((z-z_1)\log(z-z_1) \\
&\qquad\qquad -(z-z_2)\log(z-z_2)+(z_1-z_2)\Big)
\end{aligned}$$

Expressions for the Cauchy-Coordinates. We provide expressions for the Cauchy-Green coordinates and their limits on the edges. Given a closed polygon in counterclockwise orientation with vertices z_j and some point z in the interior or the exterior of the domain the Cauchy-Green coordinate of the vertex z_j is given by the expression

$$C_j(z) = \frac{1}{2\pi i}\left(\frac{B_{j+1}}{A_{j+1}}\log\left(\frac{B_{j+1}}{B_j}\right) - \frac{B_{j-1}}{A_j}\log\left(\frac{B_j}{B_{j-1}}\right)\right)$$

where $B_j = z_j - z$ and $A_j = z_j - z_{j-1}$.

Note that for a point z on an edge of the polygon the limit is different when approaching it from the interior and the exterior of the domain (but in both cases exists).

The reason for this discontinuity is the branch of the logarithm. When approaching an edge $e_j = (z_j, z_{j+1})$ from the interior of the domain the limit of $\log\left(\frac{B_{j+1}}{B_j}\right)$ is $\left(\log\frac{|B_{j+1}|}{|B_j|} + \pi i\right)$, while when approaching from the exterior of the domain the limit becomes $\left(\log\frac{|B_{j+1}|}{|B_j|} - \pi i\right)$. Thus, when the point z is on the edge e_j we replace the first logarithm in the expression for the coordinates with its limit (and similarly when z is on $e_{j-1} = (z_{j-1}, z_j)$ we replace the second logarithm with its limit).

References

[GV06] GUSTAFSSON B., VASILÂĂŹEV A.: *Conformal and potential analysis in Hele-Shaw cells.* Springer Science & Business Media, 2006.