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Direct Limit Volumes: Constant-Time Limit Evaluation for Catmull-Clark Solids

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Abstract

We present a novel approach for efficient limit volume evaluation on Catmull-Clark (CC) subdivision solids. Although several analogies exist between subdivision surfaces and subdivision volumes, extending Stam's limit evaluation technique from 2 to 3 dimensions is not straightforward, as irregularities and boundaries introduce new challenges in the volumetric case. We present new direct evaluation techniques for irregular volumetric topologies and boundary cells, which allow for calculating the limit of CC subdivision solids at arbitrary parameter values in constant time. Evaluation of limit points is a central aspect when using CC solids for applications such as simulation and multi-material additive manufacturing, or as a compact volumetric representation scheme for continuous scalar fields. We demonstrate that our approach is faster than existing evaluation techniques for every topological configuration or target parameter (u,v,w) that requires more than two local subdivision steps.

CCS Concepts
•***Computing methodologies*** → ***Volumetric models;*** •***Mathematics of computing*** → *Geometric topology;*

1. Introduction

Subdivision surfaces are the dominant representation scheme for 3D models in the entertainment industry. A conceptually infinite refinement generates the so-called *limit* — the smooth surface of the geometry — from a mesh of discrete control points. Iteratively applying the subdivision rules to calculate the limit is obviously inefficient. Stam overcame this deficiency with his groundbreaking algorithm [Sta98] for evaluating limit points in constant time. When processing subdivision surfaces, efficient limit point evaluation is vital. Therefore, Stam's algorithm decisively contributed to the wide acceptance of subdivision surfaces in many industries, including manufacturing. However, subdivision solids have yet to be taken up to the same extent, although they have great potential in applications such as physically based simulation and efficient data representation, e.g. for scalar fields. Volumetric subdivision control meshes can be used to store additional information, such as material parameters (stiffness/density), which can then be evaluated directly at arbitrary parameter values. This could be especially useful when using subdivision solids for multi-material 3D printing. In their survey paper of 2003, Chang et al. already documented the growing interest in and importance of subdivision solids [CQ03]. We suspect that an efficient volumetric limit evaluation algorithm could contribute to subdivision solids in a similar way as Stam's algorithm did for surfaces. Extending his ideas to the volumetric case, we encountered challenging topological configurations that require novel solutions for direct and exact evaluation. These configurations comprise volumetric irregularities, boundaries and sharp features.

Up to now, only cells in a regular neighborhood can be evaluated directly. Using the algorithm by Burkhart et al. [BHU10], evaluation only works for a small set of topologies and a limited set of parameter values (u,v,w). For all other cases, iterative subdivision has to be used or Stam's approach has to be applied multiple times.

In this paper, we investigate efficient limit evaluation for CC subdivision solids. Based on the subdivision scheme by Joy and MacCracken [JM99], we present new direct evaluation techniques for irregular volumetric structures as well as for regular and irregular volumetric boundary cells. In summary, we present a volumetric limit evaluation method for CC subdivision solids that works in constant time.

2. Related Work

Subdivision techniques have been around for more than four decades. From Chaikin's iterative refinement algorithm for creating smooth curves, to Doo's and Sabin's as well as Catmull's and Clark's approaches to create smooth surfaces [DS78, CC78], to volumetric subdivision techniques, e.g. by Joy and MacCracken [JM99], Bajaj et al. [BSWX02] and McDonnel et al. [MCQ04]. A good overview about existing subdivision schemes can be found in a survey paper by Chang et al. [CQ03].

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In order to create sharp features on otherwise smooth subdivision surfaces, Hoppe et al. presented adapted subdivision rules that create so-called *crease edges* [HDD*94]. Those rules were later extended, e.g. by Biermann et al. [BLZ00] for increased control and flexibility. Schweitzer presented a geometric approach for crease edges that is based on creating so-called *ghost/phantom points* to obtain sharp features [Sch96]. This concept was also used by others to define sharp features as well as boundaries on subdivision surfaces [Hav05,LB07]. Introducing crease edges alters the underlying basis functions required when evaluating the limit surface, leading to *crease basis functions* as shown by Kosinka et al. [KSD14].

Jos Stam presented a method for exactly evaluating the limit of Catmull-Clark as well as Loop subdivision surfaces in constant time [Sta98]. Zorin and Kristjansson developed an extended version of Stam's evaluation by incorporating piecewise smooth surfaces and boundaries [ZK02] using Biermann's rules [BLZ00].

Motivated by efficient limit evaluation, several approaches for simulating thin shell objects using subdivision surfaces were developed, e.g. by Grinspun et al. [GCSO99]. Burkhart et al. [BHU10] presented an FEM-like approach for Catmull-Clark solids that evaluates the volumetrical limit by applying a high number of local subdivision steps near irregular vertices.

Since direct limit evaluation is substantially more complex for solids than for surfaces, prior to our method, the the partially iterative evaluation approach by Burkhart et al. [BHU10] was the only viable way to evaluate the limit of unstructured volumetric subdivision models. In our paper, we present the first constant-time limit evaluation technique for Catmull-Clark subdivision solids.

3. Direct Limit Volume Evaluation

The structure of volumetric models inherently differs from that of surface models. While surface meshes consist of vertices, edges and faces, volumetric meshes are defined by cells, each being bounded by a set of faces. Finally, subdivision solids are evaluated on a per-cell, instead of a per-face basis.

Our direct limit evaluation technique is built upon the concepts of Stam's method for constant-time limit evaluation of Catmull-Clark subdivision surfaces [Sta98]. As in the surface case, regular topologies can be evaluated directly, using cubic B-spline basis functions. Irregular topologies have to be subdivided locally until the target point (u,v,w) can be evaluated. As for Stam's approach, the key idea is to perform all required local subdivision steps using a local subdivision matrix that does not change throughout the steps. This way, the eigenstructure of this subdivision matrix can be used to combine all local subdivision steps and evaluate the limit in constant time. However, in the volumetric case, local subdivision does not isolate irregularities as easily as in the surface case. While every edge of a closed 2-manifold surface mesh must have two neighboring faces in order to form a valid topology, edges in volumetric meshes can have arbitrarily many faces (at least two). Therefore, subdivision volumes might not only contain extraordinary vertices (EVs) – vertices with a valence other than six – but also *extraordinary edges* (EEs) – edges with a valence other than four – that require special handling. As Bajaj et al. observed, repeated subdivision steps form layered structures in the neighborhood of extraordinary edges [BSWX02]. We exploit this fact for constructing our constant-time limit evaluation technique (see Section 3.2). Furthermore, in $\mathbb{R}^3$, subdivision surfaces are typically closed, whereas subdivision solids have a boundary which requires special treatment (see Section 3.3).

3.1. The Regular Case

The generalization of bivariate B-spline surface evaluation to the trivariate case is straightforward. The definition of the trivariate basis functions $N_i(u,v,w)$ can be found in the supplemental material.

3.2. Irregular Cases

As local control meshes of a volumetric subdivision model do not form regular $4\times4\times4$-grids near irregularities, elements with an irregular topology, i.e., elements that contain extraordinary edges and/or vertices, cannot be processed using standard B-spline evaluation. Figure 1(a) shows an example of a volumetric mesh with extraordinary edges of valence 3. When evaluating the highlighted cell in Figure 1(a), it is first necessary to select an extraordinary vertex. If the cell contains more than one EE, the EV is uniquely defined as the vertex that is shared by all EEs. Otherwise, the EV is chosen as one of the endpoints of the EE. All EEs in a local control mesh must be radiating from the EV. If this is not the case, the mesh has to be subdivided once to achieve this property.

In order to evaluate the limit in constant time, the control points have to be numbered, such that the topology of the local control mesh stays consistent in each subdivision step. Additionally, the index of each point and its corresponding subdivided point must be the same in each subdivision step. In contrast to the surface case, a local volumetric mesh is not uniquely defined by the valence alone and can have an arbitrary topological configuration in the direct neighborhood of its EV. Therefore, we employ a volumetric data structure to iterate over all control points in the neighborhood of the EV in a consistent way, assigning a unique index to each control point. We start at the extraordinary vertex and traverse the local control mesh in a breadth-first manner, assigning indices first to all control points that share an edge with the EV, then to those that share a face with the EV and finally to all control points that share a cell with the EV. This way, we can assemble the block of the subdivision matrix corresponding to the vertices in the direct neighborhood of the extraordinary vertex. The exact formulation of the subdivision rules and the subdivision stencils that are used to calculate the entries of the subdivision matrix can be found in the paper by Joy and MacCracken [JM99].

In the next step, we process the remaining vertices that do not share an edge, face or cell with the EV. These vertices make up the three outer layers of the control mesh, each forming a two-dimensional control mesh with the same valence as the corresponding edge coming from the EV (see Figures 1(a) and (c)). This allows us to locally use Stam's 2D numbering scheme [Sta98].

To calculate the eigenstructure, the $M\times K$-subdivision matrix is transformed into a square matrix by removing the last $M-K$ rows, where K and M are the total number of vertices in the original and the subdivided local control mesh, respectively. The rows that are removed from the matrix correspond to the three outer layers of the subdivided control mesh.

The partitioning around the extraordinary vertex results in eight sub-elements per local subdivision step. This process together with the numbering of the sub-elements is depicted in Figure 2. In the volumetric case – unlike in the surface case – only four out of eight elements become regular ($k = 0,1,2,4$) while the other four elements may still contain EEs ($k = 3,5,6,7$).

With a naive approach, we would have to subdivide further until the point to be evaluated lies inside a fully regular cell. If the point is close to an EE but not close to the EV, the local (u,v,w) coordinate system has to be rearranged in order to continue the evaluation process. This corrupts the local numbering scheme and requires the use of multiple diagonalized eigenvalue matrices, resulting in additional matrix-matrix multiplications. Fortunately, the sub-elements along the EEs are defined by a control mesh with a layered structure that allows for direct evaluation with some ingenuity.

While fully irregular constellations as in Figures 1(a) and (c) feature multiple EEs in a single cell, layered control meshes consist of four layers of an irregular 2D control mesh as in Figures 1(b) and (d). Within each layer the valence of the EE is always equal to the valence of the 2D EV. Similar to Bajaj et al.'s approach for their MLCA subdivision scheme [BSWX02], we construct the trivariate basis functions using a tensor product of the bivariate subdivision basis functions $\varphi(u,v)$ for the corresponding two-dimensional configuration and a regular cubic B-spline basis function $N_i(w)$. The resulting trivariate basis function $N_{i,j}(u,v,w)$ corresponds to the j-th control point in layer i of the local control mesh, with K denoting the number of control points in each layer.

$$N_{i,j}(u,v,w) = \varphi_j(u,v)N_i(w) \qquad i = 1,\dots,4,\ j = 1,\dots,K \tag{1}$$

Similar to the two-dimensional case, the number of local subdivision steps required for the evaluation is defined by

$$n = \lceil \min\{-\log_2(u),\ -\log_2(v),\ -\log_2(w)\} \rceil .$$

As we do not have to rearrange the (u,v,w) coordinate system, the required local subdivisions are performed using a single diagonal matrix Λ^{n-1} containing the eigenvalues of the subdivision matrix.

$$C_n = \bar{\mathbf{A}}\mathbf{V}\Lambda^{n-1}\mathbf{V}^{-1}C_0 \tag{2}$$

Given the basis functions and the eigenstructure of the subdivision matrix, we can evaluate the volumetric limit points as

$$p(u,v,w) = C_0^T\mathbf{V}^{-T}\Lambda^{n-1}\mathbf{V}^T\bar{\mathbf{A}}^T\mathbf{P}_k^T\mathbf{N}\left(\phi_{k,n}(u,v,w)\right) \tag{3}$$

$$=: C_0^T\varphi(u,v,w), \tag{4}$$

with a volumetric picking matrix $\mathbf{P}_k$ defined analogously to the 2D case for $k = 0,\dots,6$, and a transformation $\phi_{k,n}$, which maps (u,v,w) onto the local parameter space of sub-element k at subdivision level n. We call $\varphi(u,v,w)$ the *trivariate subdivision basis functions*.

3.3. Boundary Elements

Boundary cases that usually do not arise for subdivision surfaces are omnipresent in volumetric subdivision models. As the topology of cells on the boundary of the control mesh differs significantly in comparison to interior cells, the evaluation of the corresponding elements must be examined closely. In the following, we present a method for direct evaluation of boundary elements. We make use of so-called *crease basis functions* derived from the geometric concept of *ghost/phantom points* [Sch96, Hav05].

Figure 3 shows four distinct boundary cases that have to be handled individually. In the first case, depicted in Figure 3(a), the top layer represents the boundary. The mesh features a $4\times4\times3$ configuration. As stated above, boundary faces in the trivariate case can be treated analogously to crease edges in the bivariate case. Thus, the boundary can be represented by a corresponding 2D configuration of crease edges (see Figure 3(c)). For direct evaluation of the highlighted boundary element, the corresponding regular B-spline basis function is substituted by the crease basis function C. Each of the three individual crease functions corresponds to one layer of the control mesh, where the first function, starting at value 1, corresponds to the boundary layer. The new boundary basis functions $B_i(u,v,w)$ then write

$$B_i(u,v,w) = N_{\lceil i/16\rceil}(u)\ N_{\lceil i/4\rceil\%4}(v)\ C_{i\%4}(w). \tag{5}$$

The second case, depicted in Figure 3(b), contains an extraordinary edge orthogonal to the boundary. Besides its boundary, the mesh features the same layered configuration as the mesh in Figure 1(d). Therefore, we perform the direct evaluation similarly to Case (1) but use irregular 2D subdivision basis functions in u and v, combined with the crease basis functions in the w dimension. The corresponding boundary basis functions $B_{i,j}(u,v,w)$ write

$$B_{i,j}(u,v,w) = \varphi_j(u,v)C_i(w) \qquad i = 1,\dots,3,\ j = 1,\dots,K. \tag{6}$$

In the third boundary case shown in Figure 3(d), the EE is included in the boundary. The 2D configuration with the corresponding crease edges is shown in Figure 3(f). Calculating the subdivision basis functions for the crease irregular 2D mesh in u,v and multiplying a regular B-spline basis function in the w dimension as in Equation (1), results in the corresponding trivariate basis functions that describe the shape of the element.

In the fourth case (Figure 3(e)), the top layer as well as the back side are covered with boundary faces. This boundary case is evaluated in analogy to the non-boundary irregular case (see Section 3.2), by employing a subdivision matrix to mathematically perform n local subdivision steps. After those the cell with the target point corresponds to one of the boundary Cases (1) - (3) and we can perform our direct evaluation accordingly.

Volumetric limit points inside boundary cells can finally be evaluated using Equation 3 and replacing N with the obtained boundary basis functions B. If a hexahedral cell contains boundary faces on opposite sides, the control mesh has to be subdivided once. The resulting cells automatically form one of the four boundary cases presented here. Additionally, our method supports crease edges on the boundary of the model. Again, it is assumed that only edges radiating from the EV are defined as crease edges.

In addition to the 3D positions of the limit points, the *limit values* of scalar properties stored per control point can be evaluated as well. As in the two-dimensional case, our method also supports the calculation of derivatives.

4. Results

With our method, we are able to efficiently evaluate CC subdivision solids at arbitrary parameter values. We measured the time required for the main parts of our volumetric evaluation algorithm for three different topologies. These include an irregular interior cell, a boundary cell of Case (4) (see Section 3.3) and a corner cell.

Our algorithm consists of the following three steps: construction of the subdivision matrix, calculation of its eigenstructure and evaluation with the subdivision basis functions. The first two only have to be performed once for all distinct topologies and thus can be precomputed. The third step represents the actual evaluation and consists of four matrix-matrix multiplications and the exponentiation of a diagonal matrix. Table 1 lists the relative and absolute time required for each step. Since calculating the subdivision matrix and its eigenstructure accounts for a major part of the computational cost, precomputation significantly speeds up the evaluation process.

For the non-direct approach by Burkhart et al. [BHU10], the computational cost increases linearly with the number of subdivisions required for the evaluation. In contrast, we are able to evaluate the limit in constant time with our direct evaluation method. As shown in Table 2, performing the subdivisions is faster for a small number of subdivision steps. However, beyond a certain break-even point, our direct method starts to outperform this approach (see Table 1). Table 2 summarizes the inter-dependencies of target parameters u, v, w, number of subdivision steps and required rearrangements for a non-direct limit evaluation approach. As can be seen, our approach evaluates all limit points in around 72 milliseconds.

We assessed our method using the three example models shown in Figure 4. We evaluate their limit at an increasing number of regular sample points (motivated by applications such as slicing e.g. for 3D printing), as well as Gauss-Legendre quadrature points (motivated by the application of subdivision solids in finite element simulations). Figure 5 shows the results of our performance measurements for regular sampling. As can be seen, our approach scales linearly with the number of sample points, evaluating every single limit point in constant time, while the non-direct evaluation technique shows non-linear performance. However, when only performing one or two subdivisions, Burkhart's methods is still faster.

5. Conclusions and Future Work

We presented a novel approach for efficient, constant-time limit evaluation of Catmull-Clark subdivision solids. Our direct evaluation technique can handle all possible topologies in a manifold volumetric mesh that subdivide into solely hexahedra after one subdivision step. The approach has been shown to be more efficient than the iterative approach for more than two local subdivision steps. For volumetric subdivision, irregularities cannot be isolated as easily by local refinement as for subdivision surfaces. However, we demonstrated that layered structures, which can be evaluated, are formed after conceptually performing n local subdivision steps. Furthermore, we make use of crease basis functions to also evaluate the limit volume for boundary cells in constant time. To that end, we derived the trivariate subdivision basis functions for irregular volumetric topologies which enable us to evaluate the limit in constant time without explicitly computing subdivision steps.

In the future, we want to extend our method to a broader class of polyhedra, i.e. those that do not result in hexahedra after one subdivision step, as well as to other volumetric subdivision schemes. Furthermore, we plan to apply our proposed algorithm to applications such as physically based simulation and volumetric data representation and analyze its strengths and weaknesses.

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