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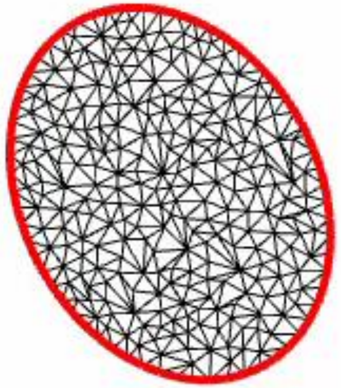

Tutorial:

Inverse Computational Spectral Geometry

Simone Melzi   Luca Cosmo   Emanuele Rodolà   Maks Ovsjanikov   Michael Bronstein

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Today’s schedule

- **I** Introduction and motivations 15:00 – 15:35
- **II** Discrete spectral geometry 15:40 – 16:15
- Coffee break 

16:20 – 16:40
- **III** Shape from spectrum and applications 16:40 – 17:15
- **IV** Localization and open problems 17:20 – 17:55

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Tutorial:

Inverse Computational Spectral Geometry

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Outline

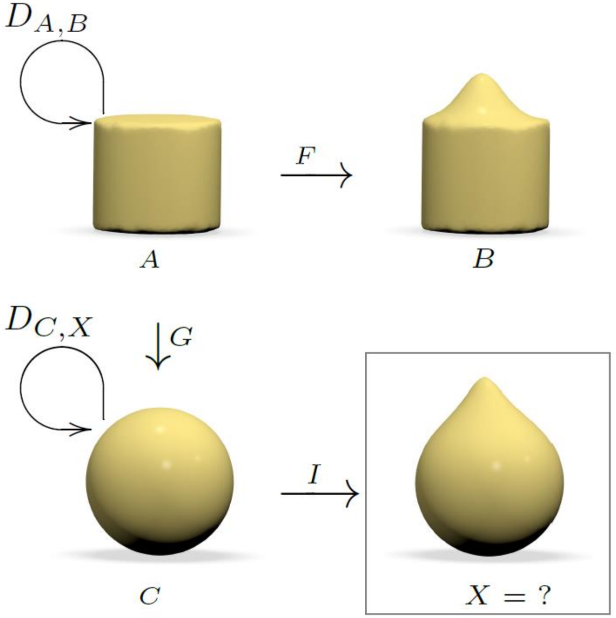
- **Motivations and Historical overview**
- **Can one hear the shape of a drum?**
- **Inverse spectral problems**
- **Teaser applications**

Shape-from-metric

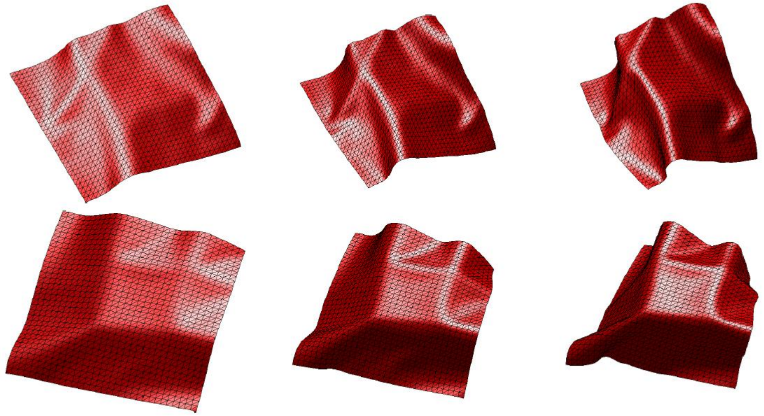
Chern et al 2018

Borrelli et al 2012

Shape-from-operator



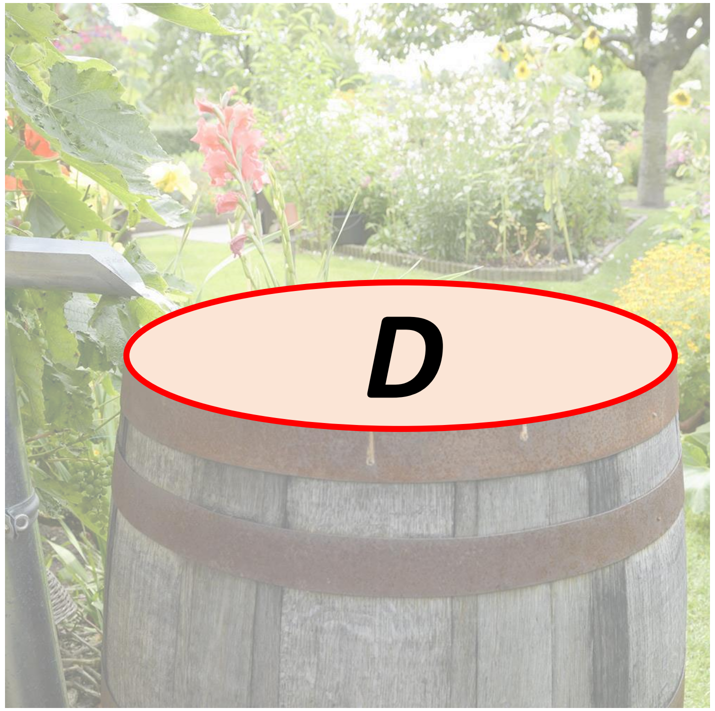
Boscaini et al 2014



Corman et al 2017

Wave equation

Wave equation



Wave equation

The wave equation for the height $f(x, y, t)$ of the water at point *(x,y)* after time *t*:

Wave equation

The wave equation for the height $f(x, y, t)$ of the water at point *(x,y)* after time *t*:

$$\Delta f = -\frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}$$

speed of sound
in the fluid

First-order approximation of the motions under consideration.

Vibrating membrane equation

The wave equation for the normal motion $f(x, y, t)$ of a vibrating membrane («drum»):

$$\Delta f = -\frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}$$

speed of sound
in the membrane

First-order approximation of sounds in a flat object.


SCI
HD

Why the eigenvalue problem?

$$\Delta f = -\frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}$$

To solve for *f*, we need only consider product functions:

$$f(x,y,t) = \phi(x,y)h(t)$$

spatial component

temporal component

Why the eigenvalue problem?

$$\Delta f = -\frac{\partial^2 f}{\partial t^2}$$

$$f(x, y, t) = \phi(x, y)h(t)$$

Laplacian eigenfunction

oscillating functions with frequency λ

$$\Delta \phi h = -\frac{\partial^2 \phi h}{\partial t^2}$$

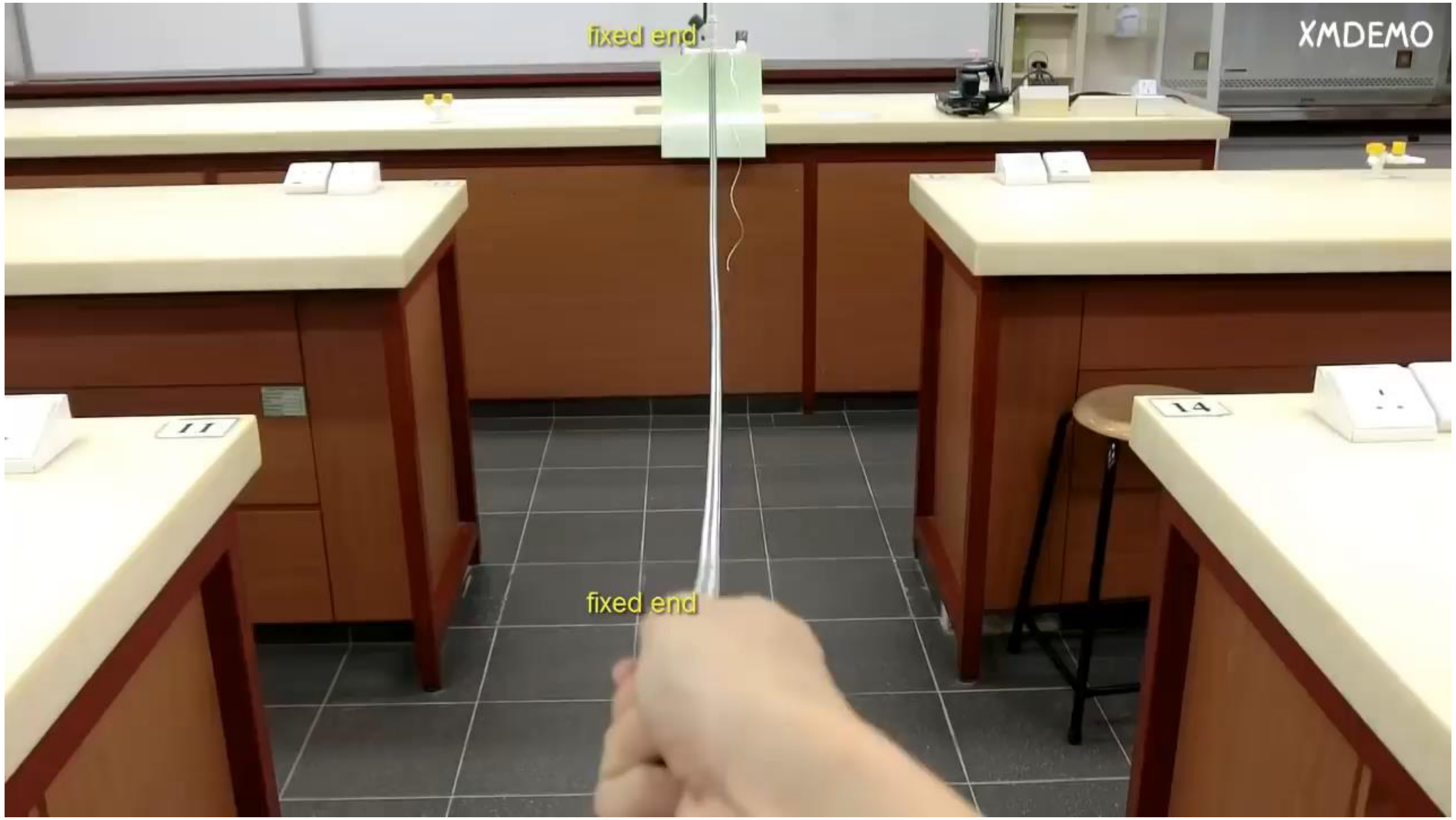
$$\frac{\Delta \phi}{\phi} = -\frac{h''}{h}$$

$$\lambda = \lambda \quad \longrightarrow \quad -\frac{h''}{h} = \lambda \quad \longrightarrow \quad h(t) = e^{it\sqrt{\lambda}}$$

$$\Delta \phi = \lambda \phi$$

Stationary waves

Physically, the product motions $f(x, y, t) = \phi(x, y)h(t)$ are **stationary**.



Video: Chua Kah Hean, 2016

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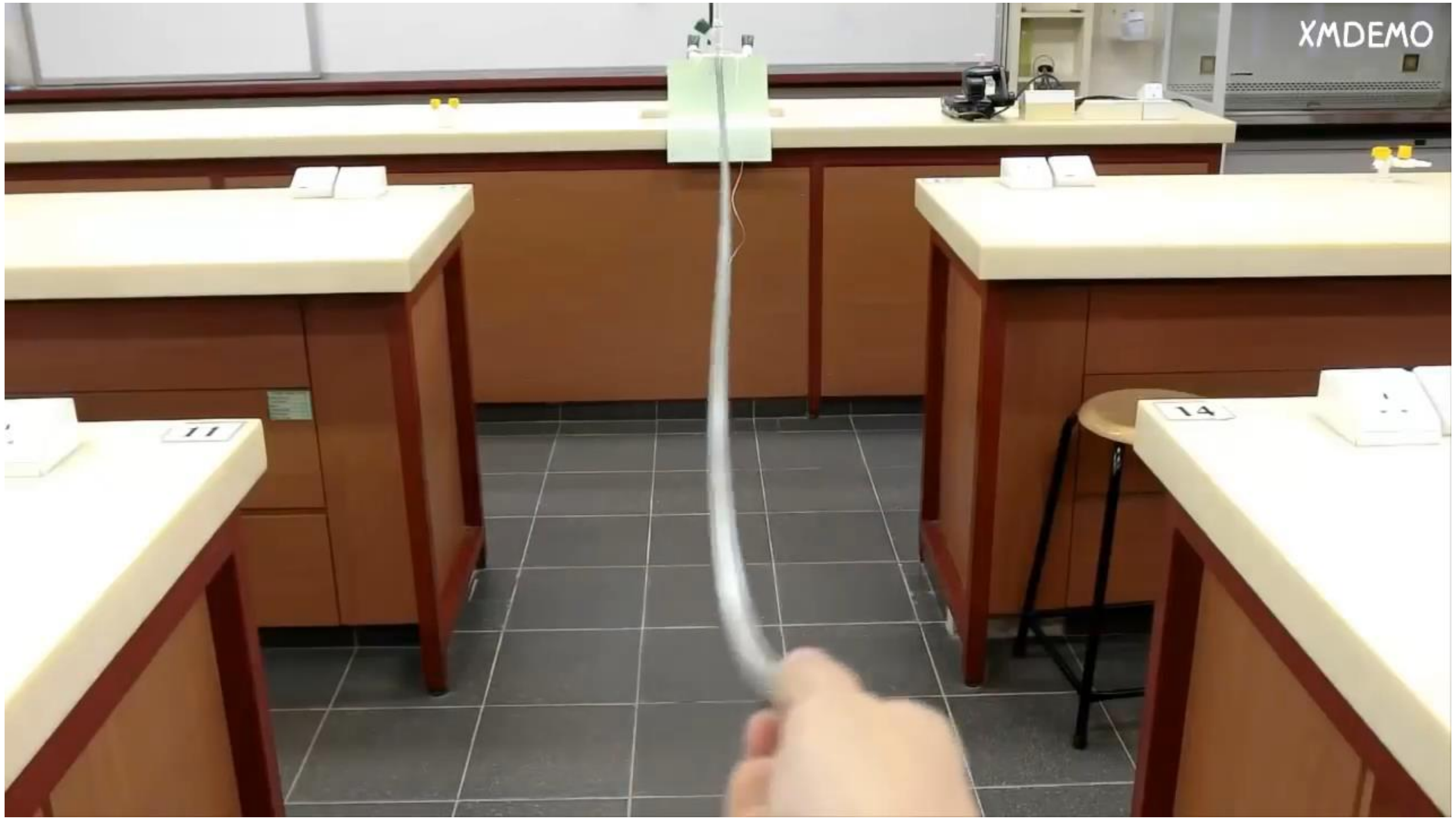


Video: Chua Kah Hean, 2016

Whispering galleries

Behavior is not always easy to grasp even on simple domains.

<u>Example:</u>
On the disk, there is high concentration along the boundary («whispering gallery effect»)

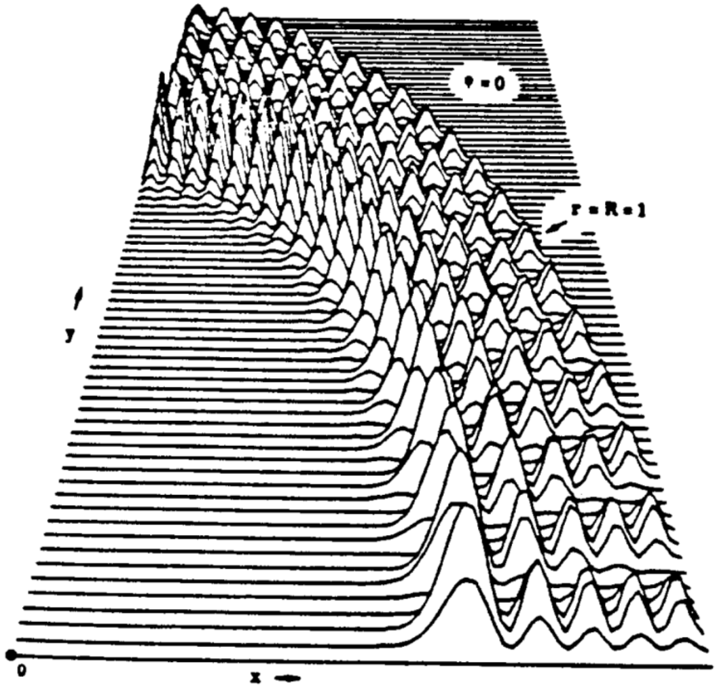


Figure: Sarnak, 1995

Voltone del Podestà, Bologna (Italy)

Computing eigenvalues

$$\Delta\phi_i = \lambda_i\phi_i$$

Very few examples where the spectrum can be determined explicitly.

«As a shocking example of our ignorance, we know nothing about regular hexagons, not even the first eigenvalue.»

[Marcel Berger, 2002]

Our drums

Direct and inverse problems

Given the (approximate) **shape** of a domain *D*,
what can I deduce about its **spectrum**?
(spectral geometry so far)

Given the (approximate) **spectrum** of a domain *D*,
what can I deduce about its **shape**?
(this tutorial)

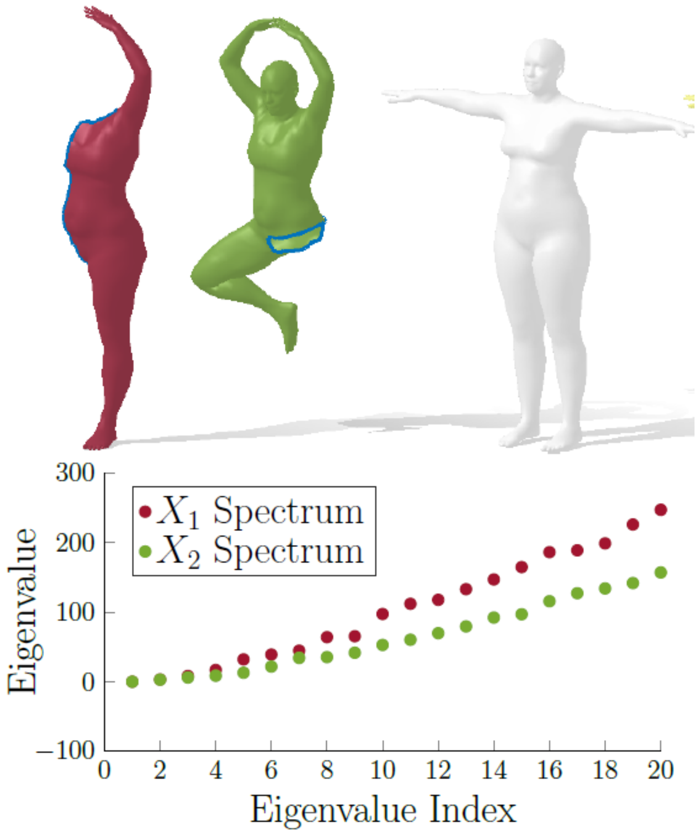
Direct problems

- Asymptotic expansion of the counting function:

$$N(\lambda) = \#\{\lambda_i \leq \lambda\}$$

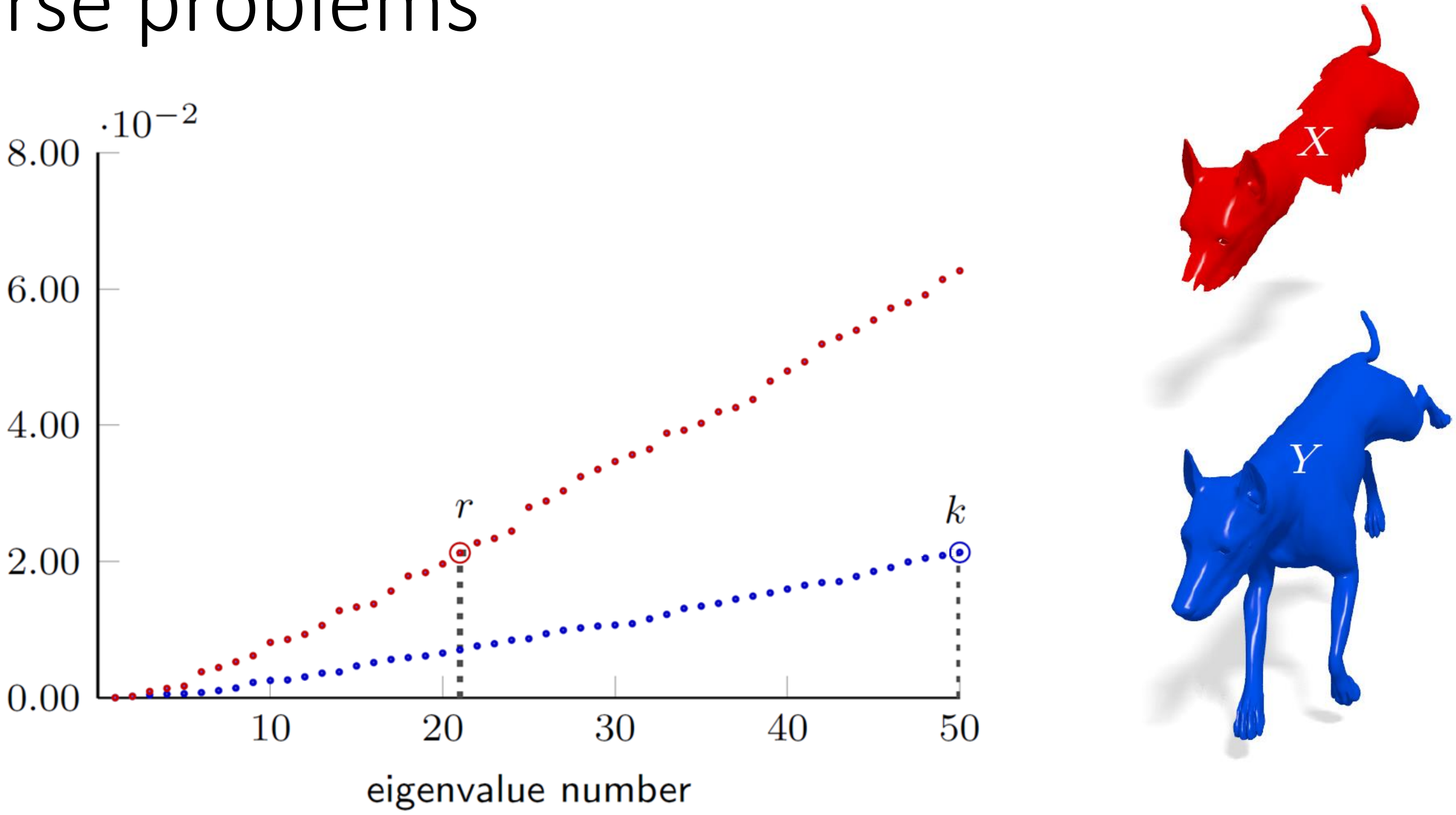
- Tight estimates of λ_1

- Relation between eigenvalues of D and those of a sub-domain $P \subset D$



[Moschella et al 2021]

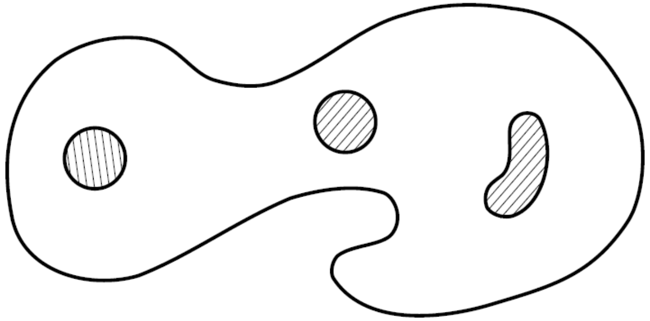
Inverse problems



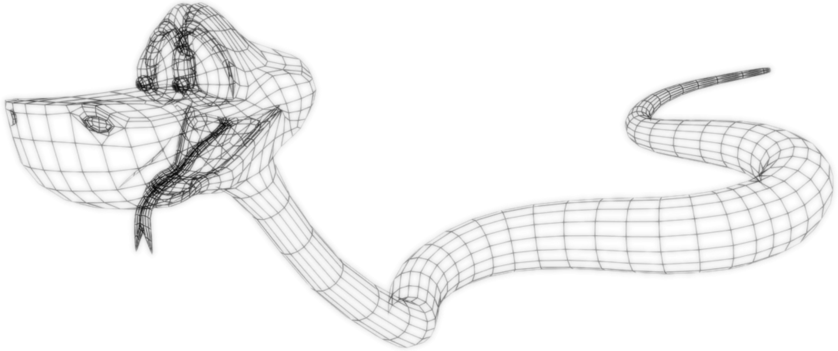
- Compute the area, perimeter, and number of holes in a shape from its eigenvalues.

Inverse problems

- Compute the area, perimeter, and number of holes in a shape from its eigenvalues.



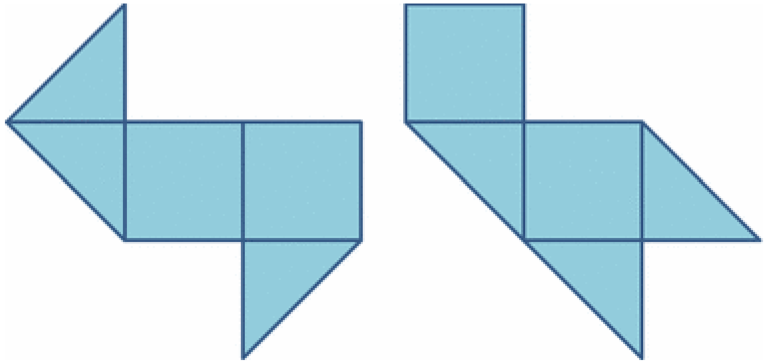
- Recover a 3D shape from its eigenvalues ~~and eigenfunctions.~~



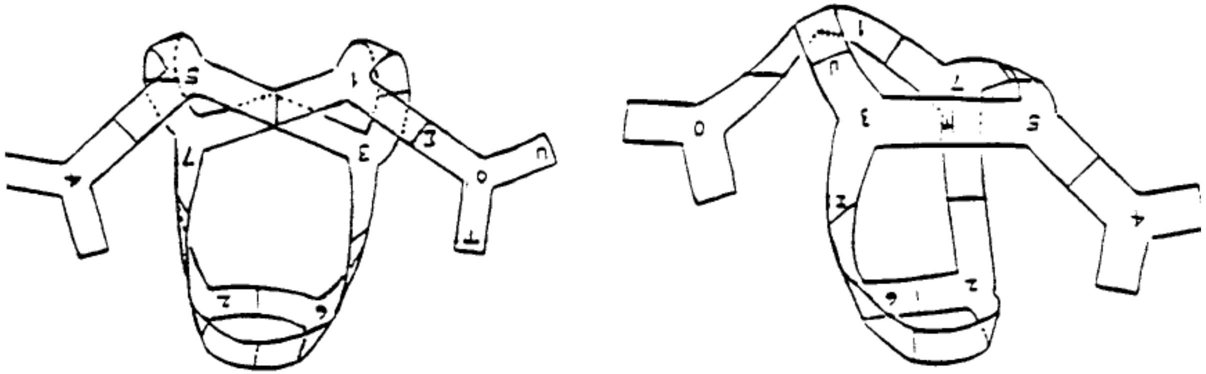
Isospectral domains

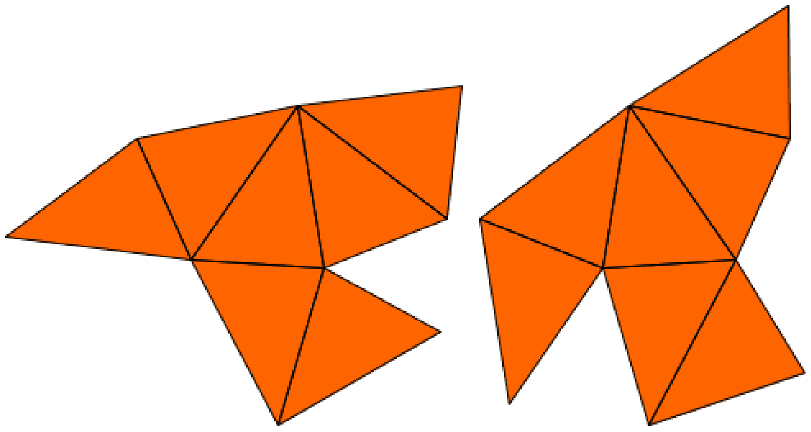
Are eigenvalues enough?

- Conjecture: yes! [Gel’fand, 1962]
- Counterexample: no! [Milnor, 1964; Gordon et al, 1992]



Except for notable exceptions (disks, spheres), in general, shapes are not fully characterized by their spectrum.



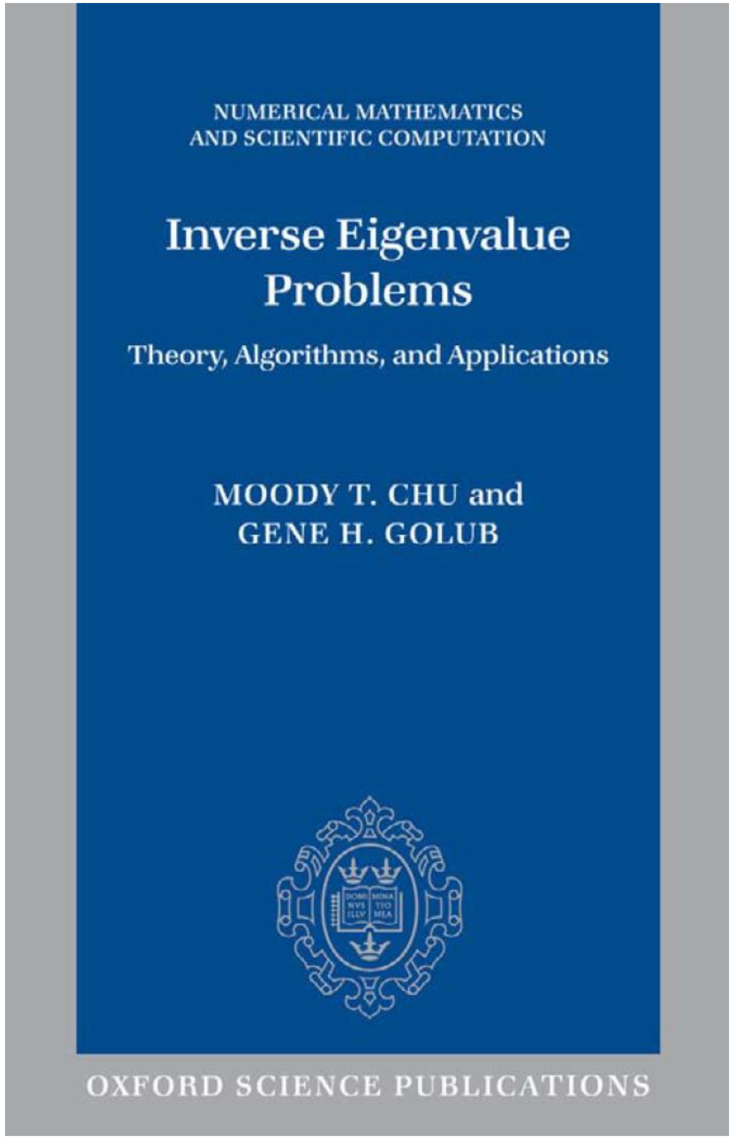


Matrix analysis

The problem has also been tackled from a purely **linear-algebraic** perspective.

These approaches assume:

- Knowledge of the matrix structure
- Partial knowledge of the eigenvectors
- Partial knowledge of the matrix entries



Matrix analysis

In 2019, Terry Tao and colleagues rediscovered a little-known result from Löwner (1934).

Eigenvectors from eigenvalues

13 August, 2019 in math.RA, paper | Tags: eigenvalues, eigenvectors, Peter Denton, Stephen Parke, Xining Zhang

Peter Denton, Stephen Parke, Xining Zhang, and I have just uploaded to the arXiv the short unpublished note "Eigenvectors from eigenvalues". This note gives two proofs of a general eigenvector identity observed recently by Denton, Parke and Zhang in the course of some quantum mechanical calculations. The identity is as follows:

Theorem 1 Let A be an $n \times n$ Hermitian matrix, with eigenvalues $\lambda_1(A), \dots, \lambda_n(A)$. Let v_i be a unit eigenvector corresponding to the eigenvalue $\lambda_i(A)$, and let $v_{i,j}$ be the j^{th} component of v_i. Then

$$|v_{i,j}|^2 \prod_{k=1; k \neq i}^{n} (\lambda_i(A) - \lambda_k(A)) = \prod_{k=1}^{n-1} (\lambda_i(A) - \lambda_k(M_j))$$

where M_j is the $n-1 \times n-1$ Hermitian matrix formed by deleting the j^{th} row and column from A.

Can it still be useful in practice?

Mathematically, the problem is beyond reach today.

Yet, in the Middle Ages, bell makers detected invisible cracks by tolling the bell.

Antonio Delli Quadri, whose family is in the bell-making business since the 14th century

"This is a complex trade that involves precise understanding of mathematics, physics, **geometry** and music"

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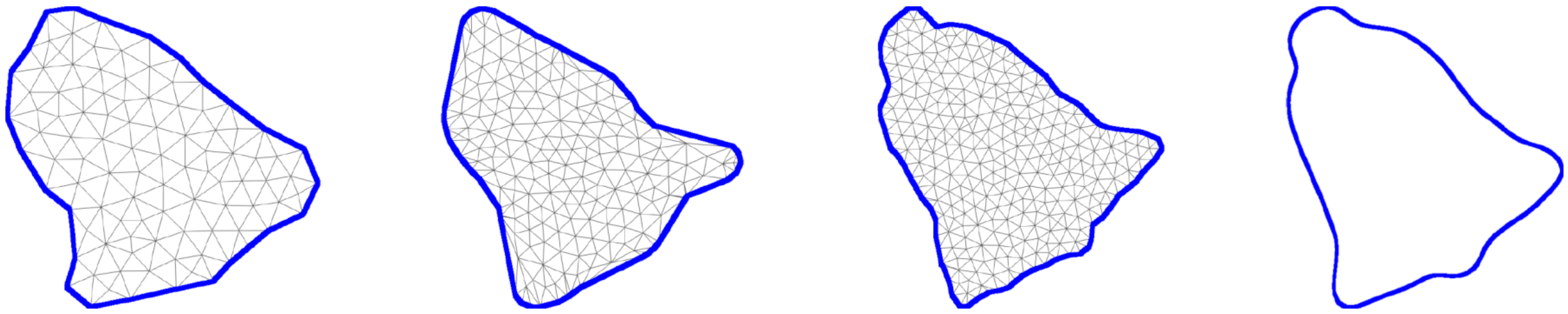

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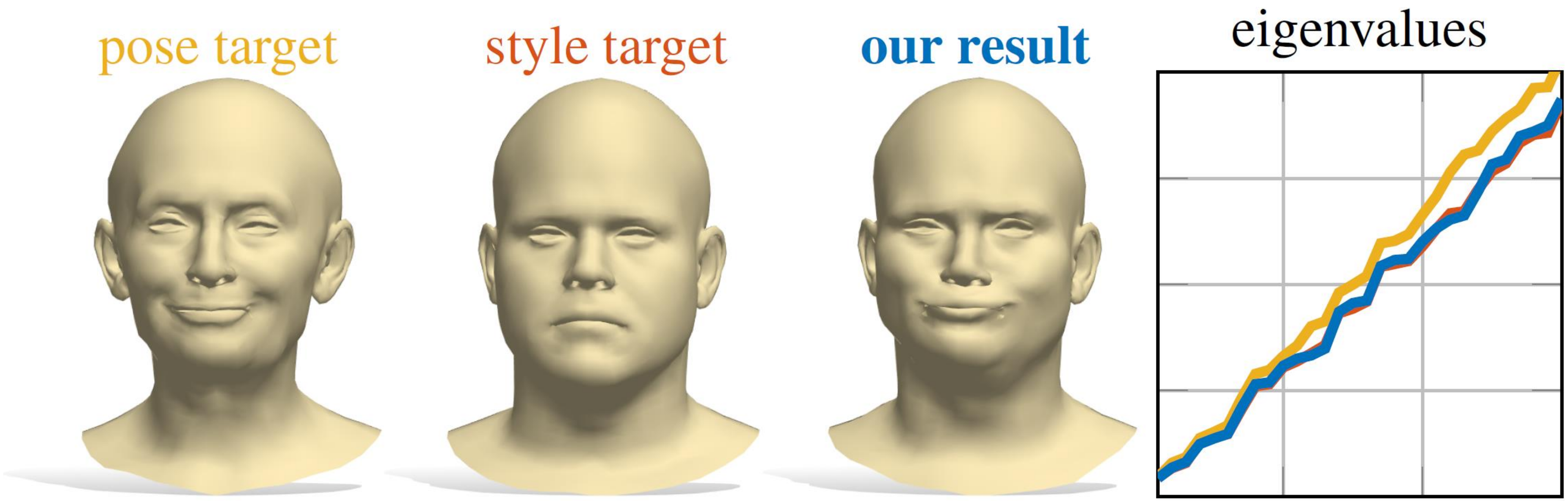

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