Chapter 5:
Vector Field Design

Hans Hagen
University of Kaiserslautern
Germany

We are now planning to design a vector field for teaching and testing purposes. The general task is to set the positions and indices of the critical points of a vector field in a plane and then define a vector field satisfying this requirements. The vector field will be given by explicit polynomial formulas.

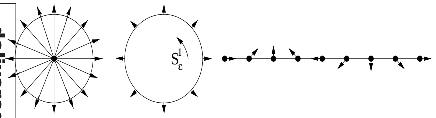
Let us start with a vector field

$$v : \Re^2 \to \Re^2$$
$$x_1 e_1 + x_2 e_2 \to v_1(x_1, x_2)\, e_1 + v_2(x_1, x_2)\, e_2 .$$

For the analysis of its topology it is necessary to analyse its zeros (critical points). An important invariant of these critical points is the Poincaré-Hopf index. Let $a \in \Re^2$ be a zero of v, i. e. $v(a) = 0$. Then we define

$$ind_a v = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{S_\varepsilon^1} \frac{v \wedge dv}{v^2}$$

It is the index of v at a, where S_ε^1 is a circle with radius ε around a.The following figure shows the geometric meaning of the index.



It is always an integer number because it counts the number of turns of the vector field on the circle with orientation. If there is no zero inside and on the circle, the index will be zero.

For our description of the field, we regard i as the generator of the rotations of the plane. We have

$$z = e_1 (x_1 e_1 + x_2 e_2) = x_1 + i x_2$$
$$z^+ = (x_1 e_1 + x_2 e_2)\, e_1 = x_1 - i x_2$$

and therefore

$$x_1 = \frac{1}{2}\left(z + z^+\right)$$
$$x_2 = \frac{1}{2i}\left(z^+ - z\right)$$

We write now

$$v : \Re^2 \to \Re^2$$
$$r = e_1 z \to v(r) = E\left(z, z^+\right) e_1$$

where E is a complex function depending on z. In general, E will not be analytic and the notation explicitly emphazises this by including z^+.

The interpretation of this formulation is that the vector field is now described as a rotation by some angle θ and a dilation by an amount $\left|E\left(z, z^+\right)\right|$ of the unit base vector e_1.

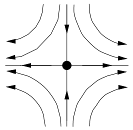
For the transformation between the cartesian description and the new description one gets

$$E(z, z^{+}) = v_1\left(\frac{1}{2}(z+z^{+}), \frac{1}{2i}(z^{+}-z)\right) - iv_2\left(\frac{1}{2}(z+z^{+}), \frac{1}{2i}(z^{+}-z)\right)$$

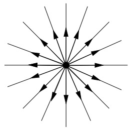
and

$$v_1(x_1, x_2) = Re(E(x_1+ix_2, x_1-ix_2))$$
$$v_2(x_1, x_2) = -Im(E(x_1+ix_2, x_1-ix_2)) .$$

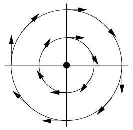
A few examples may demonstrate the effect of this different notation.



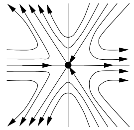
$v(r) = ze_1$



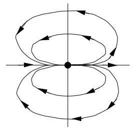
$v(r) = z^{+}e_1$



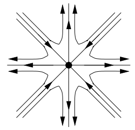
$v(r) = iz^{+}e_1$



$v(r) = z^2e_1$

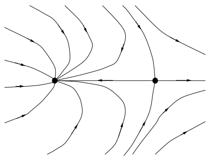


$v(r) = \left(z^{+}\right)^2 e_1$

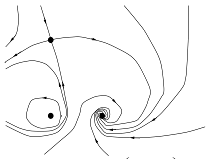


$v(r) = z^3e_1$

There are also examples with several zeros possible.



$v(r) = (z-2)z^{+}e_1$



$v(r) = iz^{+}(z-2i)\left(z^{+}-1\right)e_1$

For a comparison, we may look at the last example in cartesian coordinates :

$$\begin{aligned} v(r) &= iz^{+}(z-2i)(z^{+}-1)e_1 \\ &= i(x-iy)(x+iy-2i)(z^{+}x-iy-1)e_1 \\ &= (2x^2-2x+x^2y-2y^2+y^3)e_1 + (x^3-x^2-4xy+xy^2+2y-y^2)ie_1 \\ &= (2x^2-2x+x^2y-2y^2+y^3)e_1 + (-x^3+x^2+4xy-xy^2-2y+y^2)e_2 \end{aligned}$$

A rigorous analysis of this examples leads to the following results. Let

$$v(r) = \left(az + bz^{+} + c\right)e_1$$

be a linear vector field.

(1) For $|a| \neq |b|$, v has a unique zero at $z_0 e_1 \in \Re^2$.

(2) For $|a| > |b|$, $z_0 e_1$ is a saddle point with index -1.

(3) For $|a| < |b|$, $z_0 e_1$ is a critical point with index +1.

The critical points of index +1 can be further distinguished by the following rules completing the conventional classification :

(3a) If $Re(b) = 0$, $z_0 e_1$ will be a circle.

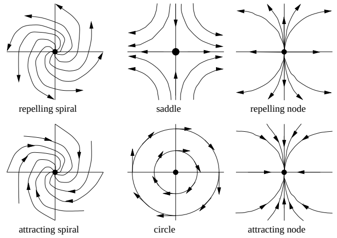
(3b) If $Re(b) \neq 0$ and $|a| > |Im(b)|$, $z_0 e_1$ will be a node.

(3c) If $Re(b) \neq 0$ and $|a| < |Im(b)|$, $z_0 e_1$ will be a spiral.

(3d) If $Re(b) \neq 0$ and $|a| = |Im(b)|$, $z_0 e_1$ will be a focus.

It may be further noticed that $z_0 e_1$ is a sink for $Re(b) < 0$ and a source for $Re(b) > 0$ in the cases (3b), (3c) and (3d).

The following figures show examples of the different types.



This result is only a reformulation of the standard classification in our Clifford algebra notation. The following result allows then the solution of our vector field design problem.
Let

$$v(r) = \left(\prod_{k=1}^{n} F_k\left(z, z^+\right)\right) e_1$$

be our vector field and the F_k be factors of E.

Then, let the vector fields

$$w_k : \Re^2 \to \Re^2$$
$$r \to F_k(r)\, e_1$$

have isolated zeros $z_1, \ldots, z_m$. These are then the zeros of v and for the indices we have

$$ind_{z_j} v = \sum_{k=1}^{m} ind_{z_j} w_k .$$

Let us now solve our design problem.
Let positions $a_1, \ldots, a_m \in \Re^2$ be given where we want to have critical points with index $i_1, \ldots, i_m < 0$. Let further positions $b_1, \ldots, b_n \in \Re^2$ be given where we want to have critical points with index $j_1, \ldots, j_n > 0$. Then we use the vector field

$$v(r) = \prod_{k=1}^{m} (e_1 z - a_k)^{i_k} \prod_{l=1}^{n} \left(z^+ e_1 - b_l\right)^{j_l} e_1 .$$

It can be shown that the vector field

$$v_k(r) = (e_1 z - a_k)^{i_k} e_1$$

has only one zero at a_k and the index is i_k by our previous considerations. The same arguments show that

$$w_k(r) = \left(z^+ e_1 - b_l\right)^{j_l} e_1$$

has only one zero at b_l with index j_l. If we put all the factors together we get the desired result if all the a_k, b_l are different.