Chapter 4: Classical Mechanics

Gerik Scheuermann
University of Kaiserslautern
Germany

4.1 Constant Force and Single Particle

A single particle is modeled as an object with position and velocity.

$$r : I \rightarrow R^3$$
$$t \rightarrow r(t)$$
$$v : I \rightarrow R^3$$
$$t \rightarrow v(t)$$

A constant force f interacts with the particle.

$$f = mg$$

Here, m is the mass of the particle and g an acceleration of the particle induced by the force.

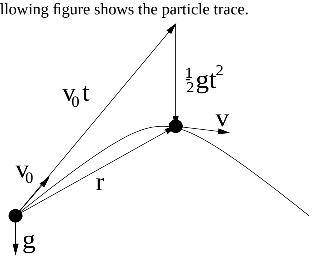
We are looking for the particle trace.

$$x : I \rightarrow R^3$$
$$x(0) = x_0$$
$$\dot{x}(0) = v(0) = v_0$$
$$\ddot{x}(t) = \dot{v}(t) = g$$

Integration gives the well-known result:

$$\dot{x}(t) = v(t) = gt + v_0$$
$$r(t) = x(t) - x_0 = \frac{1}{2}gt^2 + v_0 t$$

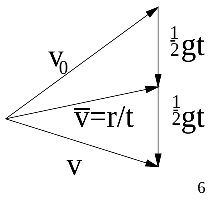
The following figure shows the particle trace.



Using the average velocity

$$\bar{v} = \frac{r}{t} = \frac{1}{2}gt + v_0$$

we can analyze this also in a v-t diagram, called **hodograph**.



We want to compute now the range of a target in direction $\hat{r}$ that has been hit by our particle starting with velocity v_0.We have

$$\frac{r}{t} = \frac{1}{2}gt + v_0$$
$$\frac{1}{t}(r \wedge r) = \frac{1}{2}(g \wedge r)t + v_0 \wedge r$$
$$\frac{1}{2}(g \wedge r)t = -v_0 \wedge r = r \wedge v_0$$
$$t = 2\frac{r \wedge v_0}{g \wedge r}$$

This is independent from the absolute value of r. If the unit direction of our target is $\hat{r}$, the time needed to hit the target is

$$t = 2\frac{r \wedge v_0}{g \wedge r} = 2\frac{\hat{r} \wedge v_0}{g \wedge \hat{r}} = 2\frac{|v_0|}{|g|}\left|\frac{\hat{r} \wedge \hat{v}_0}{\hat{g} \wedge \hat{r}}\right| .$$

The range can now be computed by

$$\frac{r}{t} = \frac{1}{2}gt + v_0$$

$$g \wedge r = gt \wedge v_0$$

$$|r| = \left|\frac{g \wedge v_0}{g \wedge \hat{r}}\right| t = \left|\frac{2\,(g \wedge v_0)\ (\hat{r} \wedge v_0)}{(g \wedge \hat{r})^2}\right| = \frac{2\,(g \wedge v_0) \bullet (v_0 \wedge \hat{r})}{|g \wedge \hat{r}|^2}$$

The term in the numerator

$$|g| 2\,(\hat{g} \wedge v_0) \bullet (v_0 \wedge \hat{r})$$

can be used to maximize the range for a fixed direction $\hat{r}$ and a fixed absolute velocity v_0^2. The general identity

$$2\,(a \wedge b) \bullet (b \wedge c) = b^2\,(a \bullet c) - a \bullet (bcb)$$

gives

$$2\,(\hat{g} \wedge v_0) \bullet (v_0 \wedge \hat{r}) = |v_0|^2\,(\hat{g} \bullet \hat{r} - \hat{g} \bullet (\hat{v}_0 \hat{r} \hat{v}_0))$$

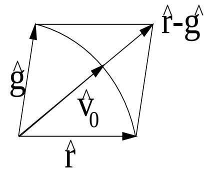
For a variation of $\hat{v}_0$, we have to maximize $-g \bullet (\hat{v}_0 \hat{r} \hat{v}_0)$. Since the brackets contain a unit vector like $\hat{g}$, we have to solve

$$-\hat{g} \bullet (\hat{v}_0 \hat{r} \hat{v}_0) = 1$$

$$-\hat{g} = \hat{v}_0 \hat{r} \hat{v}_0$$

$$-\hat{g}\hat{v}_0 = \hat{v}_0 \hat{r} .$$

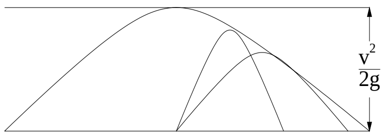
This means that the angle between g and v_0 equals the angle between v_0 and r.



With more calculations, we get a formula for the maximal range.

$$r_{\max} = \frac{|v_0|}{|g|}\frac{1}{1 - \hat{g} \bullet \hat{r}}$$

This is a paraboloid of revolution.



4.2 Constant Force with Linear Drag

A linear resistance in the direction of the velocity of a particle is also called linear drag. The force on the particle is now

$$F = mg - m\gamma v , \qquad \gamma > 0 .$$

The velocity is given by

$$\dot{v} = g - \gamma v \quad \Leftrightarrow \quad \dot{v} + \gamma v = g .$$

This can be solved by using

$$e^{\gamma t}\,(\dot{v} + \gamma v) = \frac{d}{dt}\left(ve^{\gamma t}\right) = ge^{\gamma t} .$$

For a particle with starting velocity v_0, this gives

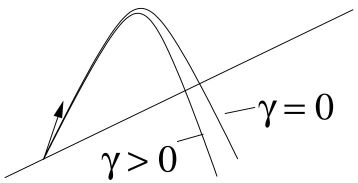
$$v(t)\,e^{\gamma(t)} - v_0 = \int_0^t ge^{\gamma s} ds = g\frac{e^{\gamma t} - 1}{\gamma}$$

$$v(t) = g\frac{\left(1 - e^{-\gamma t}\right)}{\gamma} + v_0 e^{-\gamma t} .$$

For the displacement $r = x - x_0$, we integrate to get

$$r = g\frac{e^{-\gamma t} + \gamma t - 1}{\gamma^2} + v_0\frac{1 - e^{-\gamma t}}{\gamma} .$$

We are now interested in the time a particle needs to hit the line in direction $\hat{r} = \frac{r}{|r|}$.



We use the formula for the particle trace to obtain the time of flight.

$$r = g\gamma^{-2}\left(e^{-\gamma t} + \gamma t - 1\right) + v_0\gamma^{-1}\left(1 - e^{-\gamma t}\right) \qquad | \ \gamma^2\hat{r} \wedge$$

$$\gamma^2\hat{r} \wedge r = \hat{r} \wedge g\,(e^{-\gamma t} + \gamma t - 1) + \gamma r \wedge v_0\,(1 - e^{-\gamma t})$$

$$0 = \hat{r} \wedge g\,(e^{-\gamma t} + \gamma t - 1) + \gamma\hat{r} \wedge v_0\,(1 - e^{-\gamma t}) \quad | : (\hat{r} \wedge g)$$

$$0 = (e^{-\gamma t} + \gamma t - 1) + \gamma\frac{\hat{r} \wedge v_0}{\hat{r} \wedge g}\left(1 - e^{-\gamma t}\right)$$

$$\gamma t = \left(1 - \frac{1}{2}\gamma T\right)\left(1 - e^{-\gamma t}\right), \qquad T = 2\frac{\hat{r} \wedge v_0}{g \wedge \hat{r}}.$$

T is the time needed in the case without drag. For $t < \gamma^{-1}$, we have

$$yt \approx \left(1 + \frac{1}{2}\gamma T\right)\left(\gamma t - \frac{1}{2}\gamma^2 t^2 + \frac{1}{6}\gamma^3 t^3\right) \quad | : \left(\frac{1}{2}\gamma^2 t\left(1 + \frac{1}{2}\gamma T\right)\right)$$

$$2\gamma^{-1}\left(1 + \frac{1}{2}\gamma T\right)^{-1} \approx \frac{2}{\gamma} - t + \frac{1}{3}\gamma t^2$$

$$t \approx \frac{2\left(1 + \frac{1}{2}\gamma T\right)}{\gamma\left(1 + \frac{1}{2}\gamma T\right)} - \frac{2}{\gamma\left(1 + \frac{1}{2}\gamma T\right)} + \frac{1}{3}\gamma t^2 \qquad | \ t^2 \approx T^2$$

$$t \approx \frac{T}{1 + \frac{1}{2}\gamma T} + \frac{1}{3}\gamma T^2$$

Further approximation gives

$$t \approx T\left(1 - \frac{1}{2}\gamma T\right) + \frac{1}{3}\gamma T^2$$

$$t \approx T\left(1 - \frac{1}{6}\gamma T\right)$$

The intersection can now be computed by taking the outer product of the displacement equation with $\hat{g}$.

$$r = g\gamma^{-2}\left(e^{-\gamma t} + \gamma t - 1\right) + v_0\gamma^{-1}\left(1 - e^{-\gamma t}\right) \qquad | \ \hat{g} \wedge$$

$$\hat{g} \wedge r = \hat{g} \wedge g\hat{\gamma}^{-2}\,(e^{-\gamma t} + \gamma t - 1) + \hat{g} \wedge v_0\gamma^{-1}\,(1 - e^{-\gamma t})$$

$$|r| = \frac{\hat{g} \wedge v_0}{\hat{g} \wedge \hat{r}}\left(\frac{1 - e^{-\gamma t}}{\gamma}\right)$$

With

$$\frac{1 - e^{-\gamma t}}{\gamma} \approx t - \frac{1}{2}\gamma t^2 \approx T\left(1 - \frac{1}{6}\gamma T\right) - \frac{1}{2}\gamma T^2 = T\left(1 - \frac{2}{3}\gamma T\right)$$

we have

$$|r| \approx \frac{\hat{g} \wedge v_0}{\hat{g} \wedge \hat{r}}T\left(1 - \frac{2}{3}\gamma T\right).$$

Using the range without drag

$$R = \frac{\hat{g} \wedge v_0}{\hat{g} \wedge \hat{r}}T,$$

we find finally the range as

$$|r| \approx R\left(1 - \frac{2}{3}\gamma T\right) = R\left(1 - \frac{4\gamma}{3}\frac{\hat{r} \wedge v_0}{\hat{g} \wedge \hat{r}}\right).$$

4.3 Constant Magnetic Field

The usual description for the interaction between a charged particle with charge q and mass m is

$$m\dot{v} = \frac{q}{c} v \times B \, .$$

With $\omega = \left(-\frac{q}{mc}\right) B$, we have

$$\dot{v} = \omega \times v \, .$$

Because of

$$\omega \times v = -i(\omega \wedge v) = -(i\omega) \bullet v = v \bullet \Omega, \qquad \Omega = i\omega$$

we get

$$\dot{v} = v \bullet \Omega \, .$$

Rearranging terms, this means

$$\dot{v} + \frac{1}{2}\Omega v - \frac{1}{2} v \Omega = 0 \, .$$

We are looking now for an integrating factor R with

$$\dot{R} = R\frac{1}{2}\Omega \qquad \dot{R}^{\dagger} = -\frac{1}{2}\Omega R^{\dagger}$$

This would allow

$$\frac{d}{dt}(RvR^{\dagger}) = \frac{dR}{dt} v R^{\dagger} + R\frac{dv}{dt}R^{\dagger} + Rv\frac{dR^{\dagger}}{dt} =$$
$$R\frac{1}{2}\Omega v R^{\dagger} + R\dot{v}R^{\dagger} + Rv\left(-\frac{1}{2}\Omega R^{\dagger}\right) =$$
$$R\left(\frac{1}{2}\Omega v + \dot{v} - \frac{1}{2} v\Omega\right)R^{\dagger} = 0$$

Since we have a constant magnetic field and $\Omega = \text{const.}$, we set

$$R(t) = e^{\frac{1}{2}\Omega t}$$

and find

$$R^{\dagger}(t) = e^{-\frac{1}{2}\Omega t},$$
$$R(0) = R^{\dagger}(0) = 1,$$
$$R^{\dagger}R = RR^{\dagger} = 1.$$

For the velocity, we have

$$\frac{d}{dt}(RvR^{\dagger}) = 0$$
$$RvR^{\dagger} - v_0 = 0$$
$$v = R^{\dagger} v_0 R = e^{-0.5\Omega t} v_0 e^{0.5\Omega t}.$$

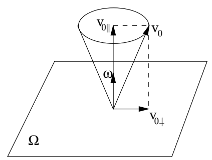
Separating v into parallel and orthogonal parts with respect to the magnetic field shows

$$v = v_{\parallel} + v_{\perp} \qquad \Omega v_{\parallel} = v_{\parallel}\Omega \qquad \Omega v_{\perp} = -v_{\perp}\Omega$$
$$R^{\dagger} v_{0\parallel} = v_{0\parallel} R^{\dagger} \qquad R^{\dagger} v_{0\perp} = v_{0\perp} R$$

This results in

$$v = v_{0\perp} R^2 + v_{0\parallel} = v_{0\perp} e^{\Omega t} + v_{0\parallel} \, .$$

This shows that the parallel velocity is constant and that the orthogonal velocity rotates through an angle Ωt in time, so v sweeps out a portion of a cone.



For the trajectory, we get

$$x - x_0 = v_{0\perp}\Omega^{-1}\left(e^{\Omega t} - 1\right) + v_{0\|}t \,.$$

With

$$r = x - x_0 + v_0 \bullet \Omega^{-1} = x - x_0 + v_0 \times \omega^{-1}$$

we have

$$r = (v_0 \bullet \Omega^{-1})\, e^{\Omega t} + v_{0\|}t = (v_0 \times \omega^{-1})\, e^{i\omega t} + (v_0 \bullet \omega^{-1})\, \omega t \,.$$

Setting

$$a = v_0 \times \omega \qquad b = v_0 \bullet \omega$$

$$|\theta| = |\omega| t \qquad \theta = |\theta|\hat{\omega}$$

one gets the standard helix

$$r(\theta) = ae^{i\theta} + b\theta$$

with

$$a \bullet \theta = 0$$