Chapter 3:
Clifford analysis

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3.1 Motivation for Differential Calculus

We know several important differential operators.

We begin with a C^1-map.

$$\varphi : \Re^3 \rightarrow \Re$$

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T \rightarrow \varphi\,(x_1, x_2, x_3)$$

We know the gradient

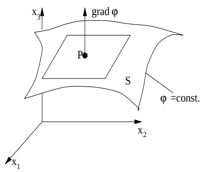
$$\text{grad } \varphi : \Re^3 \rightarrow \Re^3$$

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T \rightarrow \begin{bmatrix} \frac{d\varphi}{dx_1} & \frac{d\varphi}{dx_2} & \frac{d\varphi}{dx_3} \end{bmatrix}^T$$

with the short notation

$$\text{grad } \varphi = \nabla\varphi$$

The gradient describes the direction with the greatest rate of increase at $P = (x_1, x_2, x_3)^T$



A related operator is the directional derivative. For our map φ it is defined by

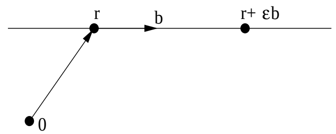
$$\varphi_b : \Re^3 \rightarrow \Re$$

$$r \rightarrow \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon}\varphi(r + \varepsilon b)$$

One also finds the notation

$$\varphi_b(r) = \nabla\varphi \bullet b$$

$\varphi_b(r)$ describes the rate of change of φ in direction b.



For a vector field

$$v : \Re^3 \rightarrow \Re^3$$

there are two important derivatives.
The divergence is the first one.

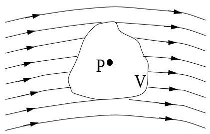
$$\text{div } v : \Re^3 \rightarrow \Re$$

$$r \rightarrow \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}$$

It has the short notation

$$\text{div } v = \nabla \bullet v$$

and measures the outflow of an infinitesimal volume V centered at P per unit volume.



The other differential operator is the curl.

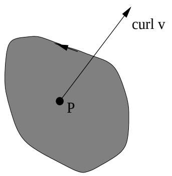
$$\text{curl } v : \Re^3 \rightarrow \Re^3$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \rightarrow \begin{bmatrix} \frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1} \\ \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \end{bmatrix}$$

It has the short notation

$$\text{curl } v = \nabla \times v .$$

The vector curl v describes the direction of a rotation axis. This axis is perpendicular to the plane where the ratio of circulation around the boundary of an area segment and the area of the segment takes its maximum.



Goal : We want to unify all this operators into one which is independent of any coordinate system.

3.2 Differential Calculus in 3D

For a coordinate invariant derivative we need the notation of reciprocal vectors in three dimensions. Let

$$\{g_1, g_2, g_3\} \in \Re^3 \subset G_3$$

be a basis. Then one defines **reciprocal vectors**

$$\{g^1, g^2, g^3\} \in \Re^3 \subset G_3$$

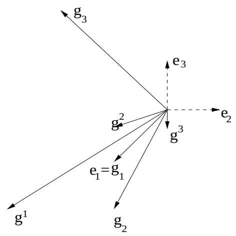
by the property

$$g_k \bullet g^l = g^l \bullet g_k = \delta_{kl} .$$

It holds

$$g^1 = \frac{g_2 \wedge g_3}{g_1 \wedge g_2 \wedge g_3} \qquad g^2 = \frac{g_3 \wedge g_1}{g_1 \wedge g_2 \wedge g_3} \qquad g^3 = \frac{g_2 \wedge g_3}{g_1 \wedge g_2 \wedge g_3}$$

The following picture shows the two sets together.



We start our construction with taking derivatives with respect to a direction. Let

$$A : \Re^3 \rightarrow G_3$$

be a multivector field. Then we call the limit

$$A_b(r) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (A(r + \varepsilon b) - A(r)) , \quad \varepsilon \in \Re, b \in \Re^3$$

the derivative of A with respect to b. It contains the same grades as A.

The following rules hold

$$A_{\beta_1 b_1 + \beta_2 b_2}(r) = \beta_1 A_{b_1}(r) + \beta_2 A_{b_1}(r)$$
$$(AB)_b(r) = A_b(r) B(r) + A(r) B_b(r)$$

The **vector derivative** is defined by

$$\partial A(r) : \Re^3 \rightarrow G_3$$
$$\partial A(r) = \sum_{k=1}^{3} g^k A_{g_k}(r)$$

where $\{g_1, g_2, g_3\}$ is a basis and $\{g^1, g^2, g^3\}$ is the reciprocal basis. The element $\partial A(r)$ has the geometric type of a product of a vector with $A(r)$. It can be shown that $\partial A(r)$ is independent of the chosen basis $\{g_1, g_2, g_3\}$.

As example, let us look at a scalar field

$$\varphi : \Re^3 \rightarrow \Re$$

and a surface $S \subset \Re^3$ with parametrization

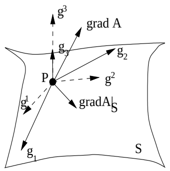
$$\mathrm{r} : \Re^2 \rightarrow S \subset \Re^3$$
$$(u_1, u_2) \rightarrow r(u_1, u_2)$$

Consider a point $P = (x_1, x_2, x_3) \in \Re^3$ and let

$$g_1 = \frac{\partial\varphi}{\partial u_1} \qquad g_2 = \frac{\partial\varphi}{\partial u_2} \qquad g_3 = g_1 \times g_2 = n$$

It holds

$$\partial\varphi = \sum_{k=1}^{3} g^k \varphi_{g_k}$$
$$= \text{grad } \varphi|_S + \text{grad } \varphi \bullet n$$
$$= \text{grad } \varphi$$



All our operators in the motivation are special cases of the vector derivative and its components. Let us look at a vector field

$$v : \Re^3 \to \Re^3$$

$$r \to v(r) = \begin{bmatrix} v_1(r) \\ v_2(r) \\ v_3(r) \end{bmatrix}$$

We get

$$\partial v = \sum_{k=1}^{3} e_k v_{e_k}$$
$$= \sum_k e_k \left(\frac{\partial v_1}{\partial e_k} e_1 + \frac{\partial v_2}{\partial e_k} e_2 + \frac{\partial v_3}{\partial e_k} e_3 \right)$$
$$= \frac{\partial v_1}{\partial e_1} e_1^2 + \frac{\partial v_2}{\partial e_2} e_2^2 + \frac{\partial v_3}{\partial e_3} e_3^2 + \left(\frac{\partial v_3}{\partial e_2} - \frac{\partial v_2}{\partial e_3} \right) e_2 e_3 + \left(\frac{\partial v_1}{\partial e_3} - \frac{\partial v_3}{\partial e_1} \right) e_3 e_1 + \left(\frac{\partial v_2}{\partial e_1} - \frac{\partial v_1}{\partial e_2} \right) e_1 e_2$$
$$= \left[\frac{\partial v_1}{\partial e_1} + \frac{\partial v_2}{\partial e_2} + \frac{\partial v_3}{\partial e_3} \right] + i \left[\left(\frac{\partial v_3}{\partial e_2} - \frac{\partial v_2}{\partial e_3} \right) e_1 + \left(\frac{\partial v_1}{\partial e_3} - \frac{\partial v_3}{\partial e_1} \right) e_2 + \left(\frac{\partial v_2}{\partial e_1} - \frac{\partial v_1}{\partial e_2} \right) e_3 \right]$$
$$= \text{div } v + i \text{ curl } v$$

3.3 Motivation for Integration

Besides differential operators, integration is essential in calculus. A very important theorem is the **divergence theorem** due to Gauss.

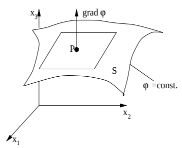
Let $V \subset \Re^3$ be a compact volume with a piecewise smooth boundary S and n the unit outward normal on S. We look at a vector field

$$v : \Re^3 \to \Re^3 .$$

We have the following relation

$$\int_V \text{div } v dV = \int_S n \bullet v dA$$

It states that for an arbitrary volume in an application the sum of the divergence in the volume is the net outflow through the surface.



Another important relation is **Stokes theorem**.
It states

$$\int_V \text{curl } v dV = \int_S n \times v dS$$

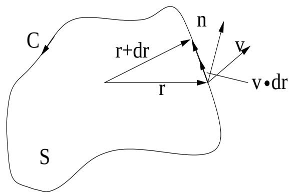
with the same notations as before and relates the sum of the curl inside the volume to the flow on the surface.

It is better known in the following case.

Let $S \subset \Re^3$ be a compact, orientable, piecewise smooth surface with oriented boundary curve C. Further, let $n \in \Re^3$ be the unit normal in accordance with the right-hand rule. Then holds

$$\int_S n \bullet \text{curl } v dA = \int_C v \bullet \text{dr} .$$

If v describes a force acting on particles, the theorem will state that the total work done on a particle traveling on C equals the integral of the curl on the surface.



3.4 Integration in 3D

We want to introduce now the integration of multivector fields. Let

$$M \subset \Re^3$$

be a smooth curve, surface or volume. Let

$$A : M \to G_3 \qquad B : M \to G_3$$

be two piecewise continous multivector fields.

Then we define the **integral**

$$\int_M A \; dX \; B$$

as the limit of

$$\lim_{n \to \infty} \sum_{k=1}^{n} A(x_k) \Delta X(x_k) B(x_k)$$

where $\Delta X(x_k)$ is a curve-, surface- or volume-segment centered at x_k with a measure in the usual Riemannian sense.

In most practical cases the set M is given by a parametrization. Let

$$r : \Re \supset J \to M \subset \Re^3$$
$$u \to r(u)$$

be a smooth curve. Then we have

$$\int_M A \; dX \; B = \int_M A(r) drB(r) = \int_J A(r(u)) du \; g(u) B(r(u)) ,$$

where $g(u) = \frac{\partial r}{\partial u}(u)$.

For a smooth surface patch

$$r : \Re^2 \supset J_1 \times J_2 \to M \subset \Re^3$$
$$(u_1, u_2) \to r(u_1, u_2)$$

we get

$$\int_M A \; dS \; B = \int_M A(r) dS(r) B(r)$$
$$= \int_{J_1 \times J_2} A(r(u)) (du_1 g_1(u) \wedge du_2 g_2(u)) B(r(u))$$

with

$$g_k(u_1, u_2) = \frac{\partial}{\partial u_k} r(u_1, u_2)$$

Analogous we have for a volume patch

$$r : \Re^3 \supset J_1 \times J_2 \times J_3 \rightarrow M \subset \Re^3$$

$$(u_1, u_2, u_3) = u \rightarrow r(u) = r(u_1, u_2, u_3)$$

the definition

$$\int_M A\ dX\ B = \int_M A(r)\, dV(r)\, B(r)$$

$$= \int_{J_1 \times J_2 \times J_3} A(r(u))\,(du_1 g_1(u) \wedge du_2 g_2(u) \wedge du_3 g_3(u)\ )\, B(r(u))$$

Two important theorems show the relation of the vector derivative and the integral.

Let $V \subset \Re^3$ be a compact oriented volume with boundary ∂V and outer unit normal n, $n^2 = 1$. Let A, B be two multivector fields on V. Then we have the **fundamental theorem for a compact volume**

$$\int_V dV\ \dot{B}\ \dot{\partial}\ \dot{A} = \int_{\partial V} dS\ B\ n\ A$$

where the dots stand for taking the derivative of both fields.

Let us examine this for a vector field

$$v : \Re^3 \rightarrow \Re^3$$

We have

$$\int_V dV \partial v = \int_{\partial V} dS n v$$

$$\int_V dV\,(\partial \bullet v + \partial \wedge v) = \int_{\partial V} dS\,(n \bullet v + n \wedge v)$$

$$\int_V dV\,(\partial \bullet v) + i \int_V dV\,(\partial \times v) = \int_{\partial V} dS\,(n \bullet v) + i \int_{\partial V} dS\,(n \times v)$$

We see by comparing the parts of different grades

$$\int_V dV \text{ div } v = \int_{\partial V} dS\ (n \bullet v)$$

the **divergence theorem from Gauss** and

$$\int_V dV \text{ curl } v = \int_{\partial V} dS\,(n \times v)$$

the **theorem of Stokes** for volumes.

If we start with a compact oriented surface $S \subset \Re^3$ with boundary ∂S and a unit normal n, $n^2 = 1$, we can prove the **fundamental theorem for a compact surface**

$$\int_S dS\ \dot{B}\,(n \times \dot{\partial})\,\dot{A} = \int_{\partial S} B dr A\ .$$

If we analyse it for the vector field v, we get

$$\int_S dS\,(n \times \dot{\partial})\,\dot{v} = \int_{\partial S} dr\ v$$

$$\int_S dS\,(n \bullet\ (\partial \times v)\,) + \int_S dS\,(\,(n \times \dot{\partial})\ \bullet \dot{v}) = \int_{\partial S} dr \bullet v + \int_{\partial S} dr \times v$$

which we may identify as the **theorem of Stokes**

$$\int_S dS\,(n \bullet\ (\partial \times v)\,) = \int_{\partial S} v \bullet \mathrm{dr}$$

and the equation

$$\int_S dS\,(\,(n \times \dot{\partial}) \bullet \dot{v}) \;=\; \int_{\partial S} dr \times v$$

which is not so well-known.

In this way we see that Clifford analysis also helps to unify important theorems from integration theory for applications.