Chapter 2: Geometry with Clifford Algebra

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2.1 Projections and Reflections

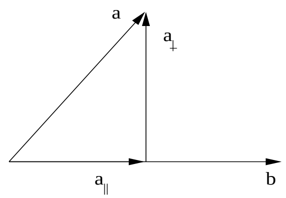
The product

$$ab = a \bullet b + a \wedge b$$

contains all the information about the relative directions of a and b. A division by b gives

$$a = (a \bullet b)\, b^{-1} + (a \wedge b)\, b^{-1}$$
$$a = a_{\|} + a_{\perp} .$$

This is a separation of the parallel and orthogonal part of a with respect to b.



If we take a 2-blade

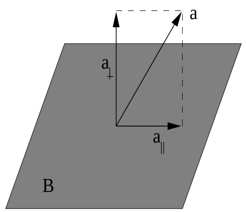
$$B = b_1 \wedge b_2$$

instead of b we get

$$aB = a \bullet B + a \wedge B$$
$$a = (a \bullet B)\, B^{-1} + (a \wedge B)\, B^{-1}$$
$$a = a_{\|} + a_{\perp} .$$

because of the existing inverse for 2-blades. In general, one can divide by all elements of pure grade.

The corresponding figure is



Another important linear operation is the reflection of vectors on a plane. We describe the plane by a bivector B and assume $|B| = 1$ because we are only interested in the direction. We set

$$x' = BxB .$$

We have

$$x = x_{\|} + x_{\perp} = (x \bullet B)\, B^{-1} + (x \wedge B)\, B^{-1}$$

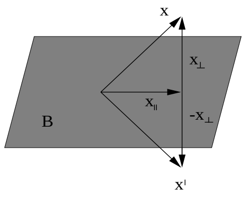
and the equations

$$x_{\|} B = B x_{\|} \quad ,$$
$$x_{\perp} B = -B x_{\perp} \quad .$$

We get

$$x' = BxB = B(x_{\|} + x_{\perp})B = x_{\|}BB - x_{\perp}BB = x_{\|} - x_{\perp}$$

so x' is the reflection of x on B.



2.2 The Exponential Function

For a multivector A the exponential is defined by

$$\exp A = e^{A} = \sum_{k=0}^{\infty} \frac{A^{k}}{k!} .$$

One might remember the matrix models for the Clifford algebra to see that this is well defined and similar to the use in the theory of ordinary linear differential equations.
We have the relations

$$e^{0} = 1$$

$$e^{A+B} = e^{A}e^{B} \qquad \text{if } AB = BA .$$

The hyperbolic cosine and sine functions are defined as

$$\cosh(A) = \sum_{k=0}^{\infty} \frac{A^{2k}}{(2k)!} = 1 + \frac{A^{2}}{2!} + \frac{A^{4}}{4!} + \dots ,$$

$$\sinh(A) = \sum_{k=0}^{\infty} \frac{A^{2k+1}}{(2k+1)!} = A + \frac{A^{3}}{3!} + \frac{A^{5}}{5!} + \dots ,$$

so we have the usual relation

$$e^{A} = \cosh(A) + \sinh(A)$$

The cosine and sine functions are defined by

$$\cos(A) = \sum_{k=0}^{\infty} (-1)^{k} \frac{A^{2k}}{(2k)!} = 1 - \frac{A^{2}}{2!} + \frac{A^{4}}{4!} - \dots$$

$$\sin(A) = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{A^{2k+1}}{(2k+1)!} = A - \frac{A^{3}}{3!} + \frac{A^{5}}{5!} + \dots$$

For any multivector I with $I^{2} = 1$ and $IA = AI$, we have

$$\cosh(IA) = \cos(A)$$

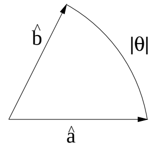
$$\sinh(IA) = I\sin(A)$$

$$e^{IA} = \cos(A) + I\sin(A) .$$

2.3 Angles

We describe one-dimensional directions by unit vectors. An **angle** is a relation between two one-dimensional directions, so we define the magnitude of the angle between two unit vectors $\hat{a}$ and $\hat{b}$ as the length of the arc on the unit circle from $\hat{a}$ to $\hat{b}$. Since the angle is measured in the plane spanned by the two unit vectors, we represent the angle as a bivector.

$$\theta = |\theta| i \qquad i = \frac{\hat{a} \wedge \hat{b}}{|\hat{a} \wedge \hat{b}|} .$$



With the exponential function of the previous section, we find the relations

$$\hat{a}\hat{b} = e^{\theta} = e^{i|\theta|} = \cos(|\theta|) + i \sin(|\theta|)$$

$$\hat{a} \bullet \hat{b} = \cos(|\theta|)$$

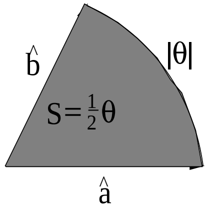
$$\hat{a} \wedge \hat{b} = i \sin(|\theta|) .$$

Elementary geometry shows

$$\frac{\text{area of sector}}{\text{arc length}} = \frac{\text{area of circle}}{\text{circumference}}$$

$$\frac{S}{|\theta|} = \frac{\pi i}{2\pi}$$

$$S = \frac{|\theta|}{2} i = \frac{1}{2}\theta$$



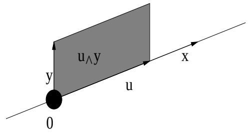
This gives an interpretation of the angle as directed plane segment, i. e. bivector which is shown in the figure.

2.4 Lines in 3D

Let $u \in G_3$ be a vector. The equation

$$x \wedge u = 0$$

describes a **line** through the origin in direction $\hat{u}$. The figure on the right shows that for $x \wedge u = 0$, x is on the line and that for $y \wedge u \neq 0$, y is not on the line.



A line with direction $\hat{u}$ through a point a is given by

$$(x - a) \wedge u = 0.$$

This is an implicit description of the line. It can be rewritten by introducing the bivector **moment** M defined as

$$M = a \wedge u.$$

We get

$$x \wedge u = M .$$

A multiplication with u^{-1} gives

$$Mu^{-1} = (x \wedge u)\, u^{-1} = x - (x \bullet u)\, u^{-1}$$

and with

$$\alpha = x \bullet u ,$$

we get the **parametric line description**

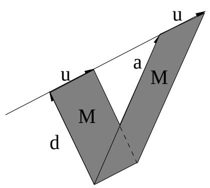
$$x = (M + \alpha)\, u^{-1}$$

With the vector

$$d = Mu^{-1} = x \wedge u \wedge u^{-1} + M \bullet u^{-1} = M \bullet u^{-1}$$

we get the **Hesse form**

$$x = d + \alpha u^{-1} .$$



It is

$$d \bullet u = \langle du \rangle_0 = \langle Mu^{-1}u \rangle_0 = \langle M \rangle_0 = 0$$

so d is orthogonal to u.
Therefore, it holds

$$|x|^2 = x^2 = d^2 + \alpha u^{-2} ,$$

and d is the distance of the line from the origin.

One may describe a line also by two points. The equation

$$(x-a)\wedge u = 0$$

says that the chords x-a is parallel to u. For two points a,b, we can define a line as all points x with the chords x-a and b-a parallel.

$$(x-a)\wedge(x-b) = 0$$

From here, we get

$$(x-a)\wedge b-(x-a)\wedge a = 0$$

$$x\wedge b-a\wedge b-x\wedge a+a\wedge a = 0$$

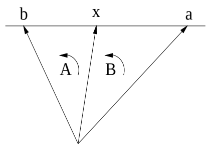
$$\frac{1}{2}(a\wedge b) = \frac{1}{2}(a\wedge x)+\frac{1}{2}(x\wedge b)$$

We set

$$B = \frac{1}{2}(a\wedge x) = |B|i$$

$$A = \frac{1}{2}(x\wedge b) = |A|i$$

where i is the unit bivector of the plane spanned by a and b. As the figure on the right shows, B and A represent triangles in this plane.



With the Jacobi identity

$$(a\wedge b)\bullet x+(b\wedge x)\bullet a+(x\wedge a)\bullet b = 0$$

and

$$a\wedge b\wedge x = 0$$

we have

$$(a\wedge b)x+(b\wedge x)a+(x\wedge a)b = 0$$

$$(A+B)x+Aa+Bb = 0$$

$$x = \frac{A}{A+B}a+\frac{B}{A+B}b\,,$$

which describes x by **barycentric coordinates**.

This description in barycentric coordinates uses really just scalar numbers since

$$x = \frac{|A|i}{|A|i+|B|i}a+\frac{|B|i}{|A|i+|B|i}b = \frac{|A|}{|A|+|B|}a+\frac{|B|}{|A|+|B|}b.$$

2.5 Planes, Spheres and Conic Sections in 3D

A **plane** with bivector direction U through a point a is given by

$$(x-a)\wedge U = 0.$$

The moment of a plane is the trivector

$$T = a\wedge U\,.$$

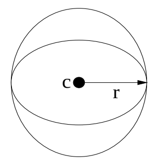
Like the line case, the vector

$$d = TU^{-1}$$

gives the distance $|d|$ of the plane from the origin.

A **sphere** with radius r and center c is defined as the set of all points $x\in\Re^3$ with

$$|x-c| = |r| \qquad\Leftrightarrow\qquad (x-c)^2 = r^2\,.$$



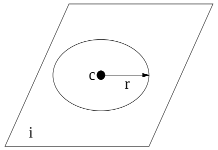
A **circle** with radius r and center c lying in the plane given by the bivector i is given by the pair of equations

$$(x-c)^2 = r^2 \qquad (x-c) \wedge i = 0.$$

A parametric equation for the circle can be given by

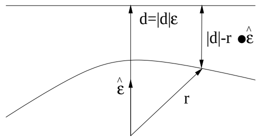
$$x - c = re^{i|\theta|} .$$

With $|i| = 1$, we need $|\theta| \in (0, 2\pi\,]$ for a unique description of all points.



A geometric definition of a **conic section** is given by the property that every point has a fixed ratio (**eccentricity**) $|\varepsilon|$ between its distance to a fixed point (**focus**) and its distance to a fixed line (**directrix**). We call r the vector from the focus to a point x on the conic section. We find from the figure with the focus at the origin

$$\frac{|r|}{|d| - r \bullet \hat{\varepsilon}} = |\varepsilon| .$$



With

$$\varepsilon = |\varepsilon|\hat{\varepsilon} \qquad l = |\varepsilon||d|$$

we get

$$\frac{|r|}{|d| - r \bullet \hat{\varepsilon}} = |\varepsilon|$$

$$|r| = |\varepsilon|\,(|d| - |r|\hat{r} \bullet \hat{\varepsilon})$$

$$|r|\,(1 + \hat{r} \bullet \hat{\varepsilon}) = |\varepsilon||d|$$

$$|r| = \frac{l}{1 + \hat{r} \bullet \hat{\varepsilon}} .$$

The standard classification of conics in two dimensions and conicoids in three dimensions is given by the following table.

Table : Classification of concis and conicoids

Eccentricity	Conic	Conicoid
$\|\varepsilon\| > 1$	hyperbola	hyperboloid
$\|\varepsilon\| = 1$	parabola	paraboloid
$0 < \|\varepsilon\| < 1$	ellipse	ellipsoid
$\|\varepsilon\| = 0$	circle	sphere

2.6 Complex numbers

A multivector in G_2 consists of a scalar, vector and a bivector part. The subset without vector part builds a subalgebra, since we have

$$z'z = (x'_1 + ix'_2)\,(x_1 + ix_2)$$

$$= (x'_1x_1 - x'_2x_2) + i\,(x'_1x_2 + x'_2x_1)$$

$$= z''$$

where i is the unit pseudoscalar of the euclidean plane.

The formulas

$$z^\dagger = x_1 - ix_2$$

$$x_1 = \frac{z + z^\dagger}{2}$$

$$x_2 = \frac{z^\dagger - z}{2i}$$

show that they can be seen as complex numbers .
The magnitude

$$|z| = \sqrt{x_1^2 + x_2^2}$$

also coincides with the usual definition for complex numbers.

We can describe a relation between complex numbers and vectors by the following simple operation

$$x = x_1 e_1 + x_2 e_2 = (x_1 + i x_2)\, e_1 = z e_1 \,.$$

We will use this in the applications to analyze vector fields by analyzing the complex number z.

2.7 Quaternions and Clifford Algebra in 3D

A multivector

$$A = \alpha + a + i\,(b + \beta)$$

in G_3 contains parts with grade 0,1,2 and 3. One may divide it in two parts of odd and even grades.

$$A = \langle A \rangle_- + \langle A \rangle_+$$

$$\langle A \rangle_- = \langle A \rangle_1 + \langle A \rangle_3 = a + i\beta$$

$$\langle A \rangle_- = \langle A \rangle_0 + \langle A \rangle_2 = \alpha + i b$$

Then, one can define the set of all odd parts G_3^- and the set of all even parts G_3^+. This second set is closed under multiplication, as may be seen from

$$\langle A \rangle_+ \langle B \rangle_+ = (\alpha + i b)\,(\gamma + i d) = (\alpha\gamma - b \bullet d) + i\,(\alpha d + \gamma b) \,.$$

This algebra of dimension four has the basis elements

$$\{1, e_1 e_2, e_3 e_1, e_2 e_3\} \,.$$

By

$$\mathrm{i} = -(e_2 e_3) \qquad \mathrm{j} = -(e_3 e_1) \qquad \mathrm{k} = -(e_1 e_2) \,,$$

one gets the quaternions invented by Hamilton.