Chapter 1:
Clifford Algebra

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1.1 Motivation

We start our considerations in the euclidean plane.
In an orthonormal basis $\{e_1, e_2\}$, we may describe a vector $v \in \Re^2$ as

$$v = v_1 e_1 + v_2 e_2$$

With the standard description as column vectors we get

$$v = v_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

If we would use square matrices, we could take

$$v = v_1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + v_2 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} v_2 & v_1 \\ v_1 & -v_2 \end{bmatrix}$$

This allows a multiplication of vectors

$$vw = \begin{bmatrix} v_2 & v_1 \\ v_1 & -v_2 \end{bmatrix} \begin{bmatrix} w_2 & w_1 \\ w_1 & -w_2 \end{bmatrix} = \begin{bmatrix} v_1 w_1 + v_2 w_2 & v_2 w_1 - v_1 w_2 \\ v_1 w_2 - v_2 w_1 & v_1 w_1 + v_2 w_2 \end{bmatrix}.$$

With a suitable choice of the remaining basis matrices we get

$$vw = (v_1 w_1 + v_2 w_2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (v_1 w_2 - v_2 w_1) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$= (v \bullet w) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (v \wedge w) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

where $\bullet$ and $\wedge$ denote the inner and outer products of Grassmann.
In this case we know them as scalar product and vector product in two dimensions.

Conclusion : We get a multiplication of vectors unifying the scalar product and the vector product in two dimensions.

In euclidean 3-space, we may use as description

$$e_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \quad e_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$v = v_1 e_1 + v_2 e_2 + v_3 e_3 = \begin{bmatrix} v_3 & 0 & v_2 & v_1 \\ 0 & v_3 & v_1 & -v_2 \\ v_2 & v_1 & -v_3 & 0 \\ v_1 & -v_2 & 0 & -v_3 \end{bmatrix}$$

For the matrix product of two vectors, we get

$$vw = \begin{bmatrix} v_1 w_1 + v_2 w_2 + v_3 w_3 & -(v_1 w_2 - v_2 w_1) & -(v_2 w_3 - v_3 w_2) & v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 & v_1 w_1 + v_2 w_2 + v_3 w_3 & v_3 w_1 - v_1 w_3 & v_2 w_3 - v_3 w_2 \\ v_2 w_3 - v_3 w_2 & -(v_3 w_1 - v_1 w_3) & v_1 w_1 + v_2 w_2 + v_3 w_3 & -(v_1 w_2 - v_2 w_1) \\ -(v_3 w_1 - v_1 w_3) & -(v_2 w_3 - v_3 w_2) & v_1 w_2 - v_2 w_1 & v_1 w_1 + v_2 w_2 + v_3 w_3 \end{bmatrix}$$

and with

$$e_1 e_2 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad e_3 e_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \quad e_2 e_3 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

this gives

$$vw = (v_1w_1 + v_2w_2 + v_3w_3)\,1 + (v_2w_3 - v_3w_2)\,e_2e_3$$
$$+ (v_3w_1 - v_1w_3)\,e_3e_1 + (v_1w_2 - v_2w_1)\,e_1e_2$$

We will see that this corresponds to

$$vw = v \bullet w + v \wedge w$$

with Grassmanns inner and outer products and that it combines the scalar and the vector product of conventional vector algebra.

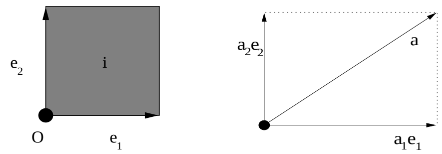
1.2 Clifford algebra in 2D

The relation between the different products in the motivation holds for different matrix representations. For a general definition in 2D we use a set of matrices $\{e_1, e_2\}$ with the following properties :

$$e_1e_2 + e_2e_1 = 0$$
$$e_j^2 = 1 \qquad \text{for } j = 1, 2$$
$$e_1e_2 \neq \pm 1$$

where 1 notes the identity matrix and $i = e_1e_2$ is called a **bivector**.

The algebra G_2 is built by real linear combinations of the basis elements $\{1, e_1, e_2, i\}$.



The i is interpreted as positive oriented area segment with area 1.

We will see a different interpretation in a later section.
The 2D-vectors are modeled by :

$$a = a_1e_1 + a_2e_2 \qquad a_1, a_2 \in \Re$$

as we could see from the right figure. A general element called **multivector** contains also a scalar and a bivector part.

$$A = a_0 1 + (a_1e_1 + a_2e_2) + a_3 i$$
$$A = A_0 + A_1 + A_2$$

A_0 describes the scalar part, A_1 the vector part and A_2 the bivector part.

The following **grade projectors** allow to deal with this parts in applications.

$$\langle\ \bullet\ \rangle_0 : G_2 \rightarrow \Re \subset G_2$$
$$A \rightarrow A_0 = a_0 1$$
$$\langle\ \bullet\ \rangle_1 : G_2 \rightarrow \Re^2 \subset G_2$$
$$A \rightarrow A_1 = a_1e_1 + a_2e_2$$
$$\langle\ \bullet\ \rangle_2 : G_2 \rightarrow \Re i \subset G_2$$
$$A \rightarrow A_2 = a_3 i$$

The **inner** and **outer products** of Grassmann can now be defined from the matrix (Clifford) product of two vectors.

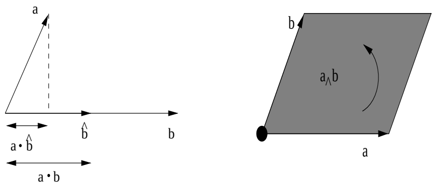
$$a \wedge b = \frac{1}{2}(ab - ba) = \langle ab \rangle_2$$
$$a \bullet b = \frac{1}{2}(ab + ba) = \langle ab \rangle_0$$

and are extended to the other grades by setting

$$A_r \wedge A_s = \langle A_rA_s \rangle_{r+s}$$
$$A_r \bullet A_s = \langle A_rA_s \rangle_{|r-s|}$$

so that general inner and outer products can be defined by linear combination of the products of the parts with pure grade.

The geometric interpretation of this products is shown in the following figures :



It is important to see that the inner product is not always the conventional scalar product. If, for example, one takes the inner product of a vector with a bivector, one will get a vector. To introduce a scalar product one defines the **reversion** operation.

$$A^{\dagger} = A_0 + A_1 - A_2$$

Then one defines the **scalar product** of multivectors A, B by

$$A * B = \langle AB^{\dagger} \rangle_0 = a_0 b_0 + a_1 b_1 + a_2 b_2 + a_3 b_3$$

which gives for vectors the usual scalar product.

The **magnitude** of a multivector is defined as usual.

$$|A| = +\sqrt{A * A} = \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2}$$

Again, we have the conventional meaning for vectors.

1.3 Clifford algebra in 3D

Geometry in three dimensions has to deal with real ratios (scalars), directed line segments (vectors), directed area segments (bivectors) and directed volumes (trivectors).

G_3 is constructed by any set of matrices $\{e_1, e_2, e_3\}$ satisfying

$$e_1 e_2 + e_2 e_1 = e_3 e_1 + e_1 e_3 = e_2 e_3 + e_3 e_2 = 0$$

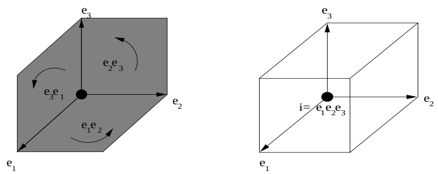
$$e_j^2 = 1 \qquad \text{for } j = 1, 2, 3$$

$$e_1 e_2 e_3 \neq \pm 1$$

and contains all real linear combinations of

$$\{1, e_1, e_2, e_3, e_1 e_2, e_3 e_1, e_2 e_3, i = e_1 e_2 e_3\}$$

A geometric interpretation is given by the following figures:



$e_1 e_2$ describes an area segment with positive orientation and area 1 in the e_1, e_2-plane.

$e_3 e_1$ stands for a positive oriented area segment in the e_1, e_3-plane and $e_2 e_3$ for a positive oriented area segment in the e_2, e_3-plane. The i is interpreted as an oriented volume segment with volume 1 and positive orientation.

The **Hodge-duality**

$$e_1 e_2 = i e_3 \qquad e_3 e_1 = i e_2 \qquad e_2 e_3 = i e_1$$

allows to describe a general multivector as

$$A = \alpha + a + i(\beta + b)$$

where

$$\alpha, \beta \in \Re, \quad a, b \in \Re^3 \subset G_3$$

Again, it is useful to define grade projectors to describe the part of a multivector with pure dimension.

$$\langle A \rangle_0 = \alpha \qquad \langle A \rangle_1 = a \qquad \langle A \rangle_2 = ib \qquad \langle A \rangle_3 = \beta$$

The **inner** and **outer products** of Grassmann are defined as

$$a \wedge b = \frac{1}{2}(ab - ba) = \langle ab \rangle_2$$

$$a \bullet b = \frac{1}{2}(ab + ba) = \langle ab \rangle_0$$

for vectors $a, b \in \Re^3 \subset G_3$.
One has again the formula

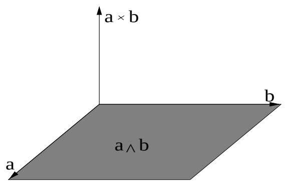
$$a \bullet b + a \wedge b = ab \ .$$

The **cross product** is related to this products in the following way

$$a \times b = i(a \wedge b)$$

and a comparison with the motivation shows that it is really the conventional cross product.

The next figure illustrates the relation between the outer and the cross product.



For inner and outer products of a vector a and a bivector B, we set

$$a \bullet B = \frac{1}{2}(aB - Ba) \ ,$$

$$a \wedge B = \frac{1}{2}(aB + Ba) \ .$$

The general inner and outer products are defined by

$$A_r \wedge A_s = \langle A_r A_s \rangle_{r+s}$$

$$A_r \bullet A_s = \langle A_r A_s \rangle_{|r-s|}$$

for elements of pure grade r and s and extended by linear composition exactly as in the 2D-case.

The **reversion**

$$A^{\dagger} = \alpha + a - i(\beta + b)$$

allows the definition of the scalar product.
The **scalar product** of two multivectors

$$A = \alpha + a + i(\beta + b),\ B = \gamma + c + i(\delta + d)$$

is defined by

$$A * B = \langle AB^{+} \rangle_0 = \alpha\gamma + a \bullet c + \beta\delta + b \bullet d$$

For the **magnitude** one sets

$$|A| = +\sqrt{A * A} = \sqrt{\alpha^2 + a^2 + \beta^2 + b^2}$$

and this is again the usual length if A is a vector.