Integral Geometry Tools for Computer Graphics

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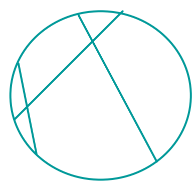
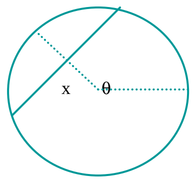
Bertrand paradox



Question: probability for the length of random chord in unit circle to exceed $\sqrt{3}$ (length of inscribed equilateral triangle). The answer depends on what is meant by random!

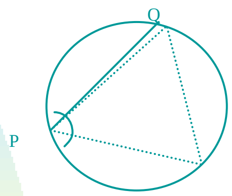
Bertrand paradox. First density



Take a random direction in the circle and then a random point in the circle. Chord has to intersect from half of the radius towards the center. Probability = 1/2

$$f_1(x,q) = \frac{dx\,dq}{2p}$$ Homogeneous and isotropic!

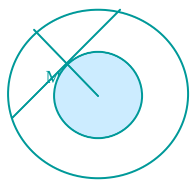
Bertrand paradox. Second density



Take two random points P,Q on the circle. Q has to be on the arc with length $\frac{p}{3}$ subtended by angle at P. Probability = 1/3

$$f_2(x,q) = \frac{dx\,dq}{p^2\sqrt{1-x^2}}$$

Bertrand paradox. Third density



Take a random point M, the chord midpoint, in the circle. M has to be in the concentric circle with radius 1/2. Probability = 1/4

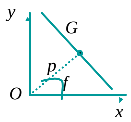
$$f_3(x,q) = \frac{x dx\,dq}{p}$$

Integral Geometry target: densities of geometric objects invariant under rotations and translations and associated measures

- densities of lines;
- densities of planes;
- densities of bodies, kinematic density;
- measures of intersections,e.g.,lines intersecting a body

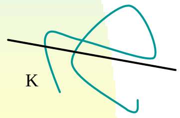
Geometric probability: quotient of associated measures, Laplace rule

2D line density



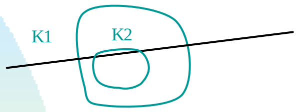
$dG = dp\,df$

If n(G) is the number of intersections of line G with curve K:



$\int n(G)dG = 2L$

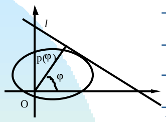
Measure of lines intersecting a convex body



Measure of lines intersecting a convex body (n(G) =2): $\int dG = L$

Probability that a line intersecting K1 also intersects K2: $\frac{L_1}{L_2}$

Support functions



- K –convex body
- O Î K – coord system origin
- l – support line
- $p(j)$ – the distance from O to l, perpendicular to the direction j

Definition. Function $p(j)$ is called the *support function* of the convex body K, related to the origin O.

L – the perimeter of convex body K

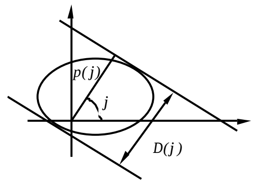
$$L = \int_0^{2p} p(j)\,dj$$

Breadth (or thickness)

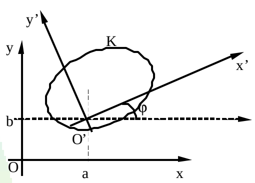
Definition. The *breadth* $D(j)$ of the convex body K in the direction j is the distance between to two parallel support lines to K, that are perpendicular to the direction j, such that the body K is in-between them.

$D(j) = p(j) + p(j+p)$.

$L = \int_0^p \Delta(j)\,dj$

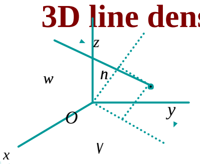


Densities of bodies

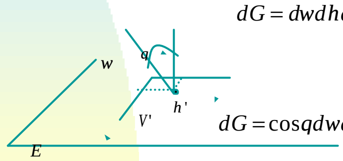


kinematic density: $dK = da\,db\,dj$

3D line density

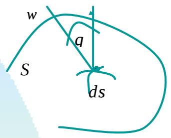


$dG = dw\,dh\,dV$



$dG = \cos q\,dw\,dh'\,dV'$

Lines intersecting a surface

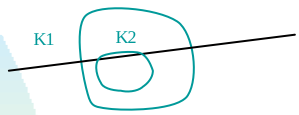


$$dG = \cos\theta \, d\omega \, ds$$

If n(G) is the number of intersections of line G with surface S: $\int n(G)\,dG = \pi A$

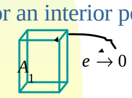
For a convex body: $\int dG = \frac{\pi}{2} A$

Probability of line intersection for a convex body interior to a second one

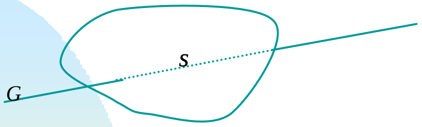


Prob. of intersection (3D): $\frac{A_1}{A_2}$

For an interior polygon: $\frac{2A_1}{A_2}$



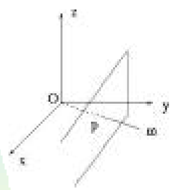
Measure of chords



$$\int s\,dG = \int_{\Omega} d\omega \int s\,dh\,dV = 2\pi V$$

Average length of chord for a convex body: $\frac{2\pi V}{\frac{\pi}{2} A} = \frac{4V}{A}$ (i.e., for a sphere: $4/3r$)

3D plane density



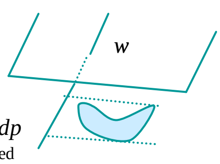
$$dE = dp\,d\omega$$

Measure of planes intersecting a convex body: Mean curvature M

For a convex polygon of perimeter L: $M = \frac{\pi L}{2}$

Parallelepiped with edges a,b,c: $M = \pi(a + b + c)$

3D thickness

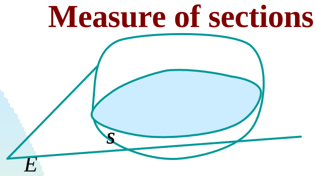


$$T(\omega) = \int_{\omega \text{ fixed}} dp$$

Measure of planes intersecting a body: $m^E(K) = \int_{S^2} T(\omega)\,d\omega$

For a convex body total thickness is equal to mean curvature M

Measure of sections



$$\int s\,dE = \int s\,dp\,d\omega = \int_{\Omega} d\omega \int_{p(-\omega)}^{p(\omega)} s\,dp = V \int_{\Omega} d\omega = 2\pi V$$

Average section of a random plane with a convex body: $\frac{\int s\,dE}{M} = \frac{2\pi V}{M}$

Optimality criteria

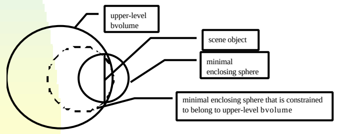
Optimality criterion for line shooting: minimization of the perimeter of the bounding volume.

Line shooting - similar to ray shooting.
Difference: the measure of lines intersecting a body is finite, while the measure of rays is infinite.

Bounding volumes & optimality criteria

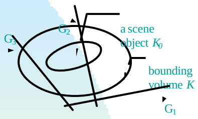
- The use of bounding volumes in CG
- Hierarchical and non-hierarchical Bvolumes
- Types of volumes (AABB, OBB, spheres, slabs)

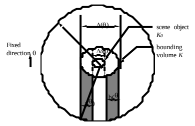
$$T = N_v * C_v + N_p * C_p$$



"Hierarchical nested-ness criterion" may yield bounding volumes that are not optimal

Bounding volume optimality criteria (ray shooting)



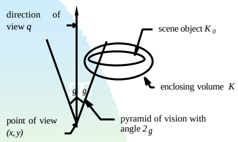


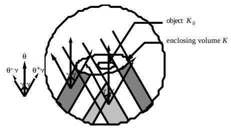
Improper intersections: $p = \frac{m_{improper}}{m_{all}}$

Theorem: Optimal bounding volume for ray shooting has minimal perimeter

Note: Importance of uniform distribution of rays

Bounding volume optimality criteria (frustum culling)





Theorem: Optimal bounding volume for frustum culling has minimal perimeter

Construction algorithms

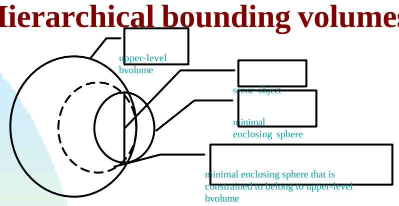
2D case

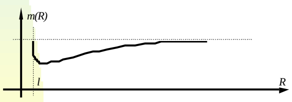
minimal perimeter bounding rectangle:
convex polygon - *O(N)*
point sets - *O(NlogN)*

3D case

Optimal bounding prisms

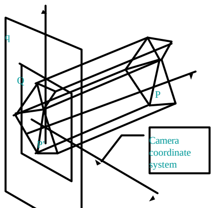
Hierarchical bounding volumes





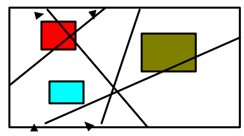
"Quality" of bounding volumes as a function of radius *R*

Polygon triangulations



Optimal triangulation – minimal perimeter triangulation

Global lines generation: from the walls of a convex bb

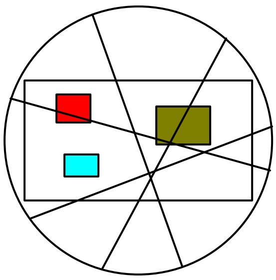


Random point on bb wall and random direction according to $\cos q \, dw$

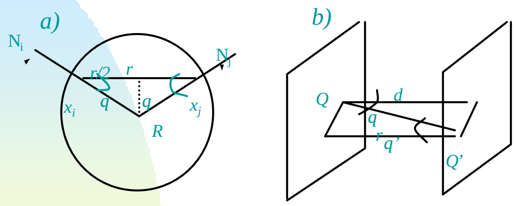
i.e., random "local" lines from the walls, taking into account all intersections.

Global lines generation: from the bounding sphere

Pairs of random points on the sphere define global lines (valid in 3D, but not in 2D!)



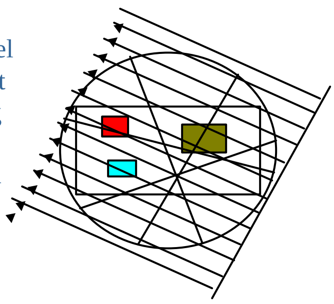
Correct and incorrect global lines generation



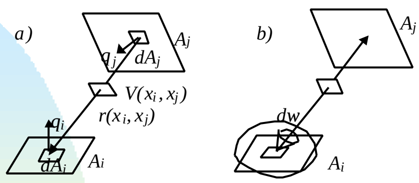
(a) Geometry for two points on sphere line generation.
(b) Geometry for incorrect line generation. Ratio of densities equal to: $\frac{\cos q \cos q'}{r^2}$

Global lines generation: from tangent plane

Bundles of parallel lines from tangent plane to bounding sphere. Average intersections with surface i $\frac{A_i}{2\Delta}$ where Δ is the bundle section (pixel area)



Geometry for Form Factors



Form Factor as area integral (a)

$$F_{ij} = \frac{1}{pA_i}\int_{A_i}\int_{A_j}\frac{\cos q_i \cos q_j}{r^2} V_{ij}\, dA_i\, dA_j$$

and hemisphere integral (b).

$$F_{ij} = \frac{1}{pA_i}\int_{\Omega}\int_{A_i} V_{ij}(x, w)\cos q\, dA_i\, dw$$

Monte Carlo Integral

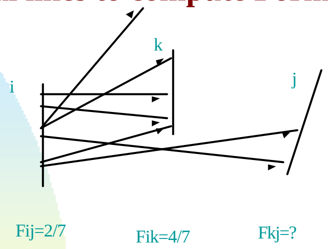
$$F_{ij} \approx \frac{1}{N}\sum_{k=1}^{N}\left(\frac{1}{pA_i}\right)\frac{V_{ij}(w_k,x_k)\cos q_k \sin q_k}{f(w_k,x_k)}$$

$$= \frac{1}{N}\sum_{k=1}^{N}\left(\frac{1}{pA_i}\right)\frac{V_{ij}(w_k,x_k)\cos q_k \sin q_k}{\frac{\cos q_k \sin q_k}{pA_i}}$$

$$= \frac{1}{N}\sum_{k=1}^{N} V_{ij}(w_k,x_k)$$

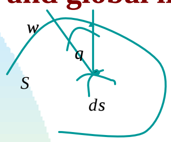
Using Monte Carlo integral with pdf $f(x,w)=\frac{\cos q}{pA_i}$ to compute Form Factor integral we obtain a sum of binary visibilities.

Local lines to compute Form Factors



Local lines from patch i distributed according to pdf $f(x,w)=\frac{\cos q}{pA_i}$ are used to compute Form Factors from i.

Relationship between Form Factor and global lines densities



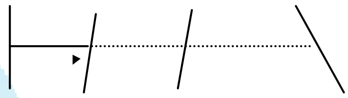
$$dG = \cos q\, dw\, ds$$

With global line parametrization from a surface:

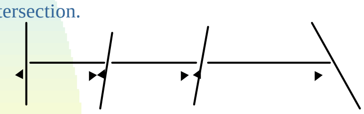
$$F_{ij} = \frac{1}{pA_i}\int_{\Omega}\int_{A_i} V_{ij}(x,w)\cos q\, dA_i\, dw = \int_{G\cap i\neq f} V_{ij}(G)\, dG$$

A global density of lines submits on each surface the "local" density corresponding to the Form Factors one.

Local and global lines

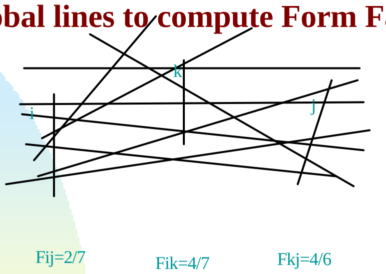


With "local" lines we can only use the first intersection.



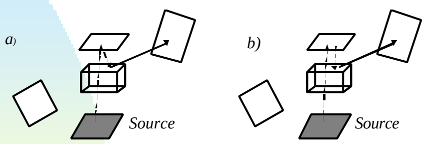
With "global" lines we can use bidirectionally all intersections.

Global lines to compute Form Factors



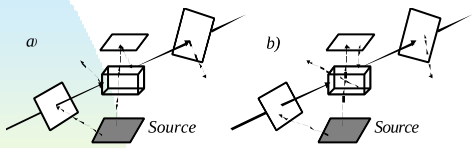
Random "global" lines can be used to compute Form Factors for all intersected patches.

Random walk generated with local lines



A local line makes advance one single path a)keeping impinging point b)sorting new exiting point

Random walk generated with global lines

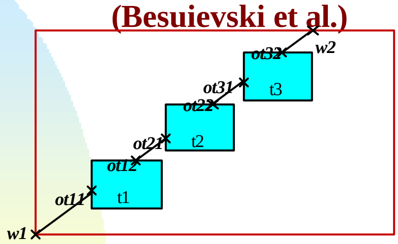


A global line makes advance several paths ad once a)idealized situation b)actual paths

Some Radiosity results

R.Martínez et al.:Multipath algorithm with bundles of lines

Global lines can be used with dynamic environments: Multiframe method (Besuievski et al.)

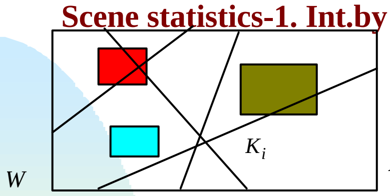


Intersection list: w1,ot11,ot12,ot21,ot22,ot31,ot32,w2

Three intersections list extracted for t1,t2 and t3:

w1,ot11,ot12,w2; w1,ot21,ot22,w2; w1,ot31,ot32,w2

Scene statistics-1. Int.by lines



$$K = \bigcup_i K_i$$

$$A(K) = \sum_i A(K_i)$$

Average number of intersections of a line crossing convex cavity W with interior bodies K_i: $n^G_{\text{int}} = \frac{2A(K)}{A(W)}$

Idem intersecting K: $n^{G*}_{\text{int}} = \frac{pA(K)}{m^G(K)}$

Scene statistics-2

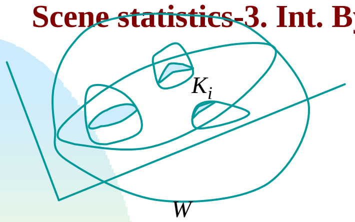
Probability of 0 intersections: $p(0) = 1 - \frac{n^G_{\text{int}}}{n^{G*}_{\text{int}}}$

Probability of i intersections: $p(i) \le \frac{A(K)}{iA(W)}$

Average length of the sum of the chords per global line: $\frac{4\sum V(K_i)}{A(W)}$

Idem per chord: $\frac{4\sum V(K_i)}{A(K)}$

Scene statistics-3. Int. By planes

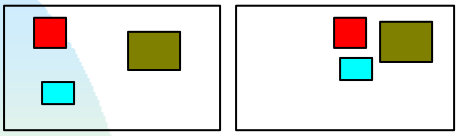


$$K = \bigcup_i K_i$$

Average number of objects in K intersected by a plane intersecting W: $n^E_{\text{int}} = \frac{\sum T(K_i)}{M(W)}$

Idem intersecting K: $n^E_{\text{int}} = \frac{\sum T(K_i)}{T(K)}$

Which scene statistics gives more useful info about the scene?



Same average number of intersections!

But different *p(i)* distribution.

Objective: given scene statistics obtain the best structuration for ray-tracing intersection. Some steps begun in Vlastimil Havran PhD.

Continuous mutual information computation

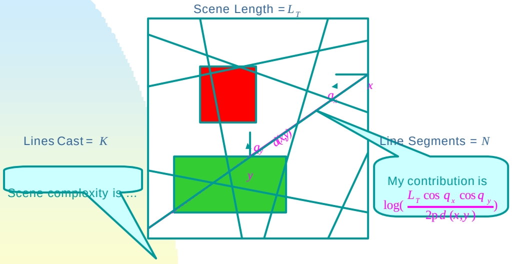
Visibility continuous mutual information is the least upper bound to the visibility discrete mutual information:

$$I_s^c = \int_{x\in S}\int_{y\in S}\frac{1}{A_T}F(x,y)\log(A_T F(x,y))dxdy$$

Cheap cost Monte Carlo computation:

$$I_s^c \approx \frac{1}{N}\sum_{k=1}^{N}\log(A_T F(x_k,y_k)) = \frac{1}{N}\sum_{k=1}^{N}\log(\frac{A_T \cos q_x \cos q_y}{pr^2})$$

Continuous mutual information computation



$$I_s^c \approx \frac{1}{N}\sum_{k=1}^{N}\log(\frac{L_T \cos q_{x_k} \cos q_{y_k}}{2pd})$$