Transformation of a Boy's Surface into a Steiner's Surface and torus eversion

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Abstract

*The part 1 of the video shows a Boy's surface and a Steiner's Roman surface which are difficult to visualize without computer **3d** graphics and demonstrates a continuous topological transformation between them. The part 2 of the video shows step by step how a two-dimensional torus with a hole can be turned inside out in R^3 by a continuous topological operation without self-intersections. The torus without a disk, Boy's surface and Steiner's Roman surface are constructed as photorealistic two-sided and front-back coloured **3d** objects.*

Categories and Subject Descriptors (according to ACM CCS):
I.3.7 [Computer Graphics]: Three-Dimensional Graphics and Realism,
I.3.5 [Computer Graphics]: Computational Geometry and Object Modeling

1. Transformation of a Boy's Surface into a Steiner's Roman Surface

Steiner's Roman surface (Jacob Steiner, 1844) and Boy's surface are two different realizations of the real projective plane (as surfaces in the R^3 space).

The real projective plane is defined as "the set that results when antipodal points of the sphere $S^2(r)$ are identified".
The parametric equations of the Steiner's Roman surface given below can be obtained applying to a sphere the *romanmap*$(x, y, z) = (xy, yz, zx)$. Boy's surface is a model of the projective plane without singularities [1] .

We can apply the **3d** modeling software „Amorphium" ("Play Inc.") in order to visualize these mathematically interesting surfaces and consider some of their topological properties.

At first, let's generate parametric 3d plots of the surfaces with help of the software „Mathematica". We use the following forms of the equations :

$$x = 2/3((\text{Cos } u \text{ Cos } 2v + \sqrt{2}\text{ Sin } u \text{ Cos } v)\text{Cos } u)/$$

$$(\sqrt{2} - \text{Sin } 2u \text{ Sin } 3v),$$

$$y = 2/3((\text{Cos } u \text{ Sin } 2v - \sqrt{2}\text{ Sin } u \text{ Sin } v)\text{Cos } u)/$$

$$(\sqrt{2} - \text{Sin } 2u \text{ Sin } 3v),$$

$$z = (\sqrt{2}\text{ Cos}^2 u)/(\sqrt{2} - \text{Sin } 2u \text{ Sin } 2v)$$

for Boy's surface and

$$x = \text{Cos } u \text{ Sin } u \text{ Cos}^2 v$$
$$y = \text{ Sin } u \text{ Sin } v \text{ Cos } v$$
$$z = \text{ Cos } u \text{ Sin } v \text{ Cos } v$$

for Steiner's Roman surface (the radius of the sphere to which the Roman map is applied is $\sqrt{2}$) correspondingly, where u and v vary from 0 to π for the both surfaces.

Moreover the number of points of every 3d plot must be the same for the both surfaces, for example, 80 plot points (PlotPoints --> 80). After that we export the surface plots from „Mathematica" as two files *.**dxf**.

Let's import one of the obtained **3d** objects (**dxf** format), „Boy's surface", to „Amorphium", a software for **3d** object modeling and topological operations with them.

Applying to this object the operation „Invert Normals" we obtain a new object. Choose the white paint colour from the palette of „Paint and Optics" menu and fill the surface with this white color; then merge this object with the source **3d** object which has „correct normals" and colour light blue by default.

We have constructed a photorealistic two-sided front-back coloured **3d** object „Boy's surface". The light blue points of this new **3d** object have „correct normals", the white points of the object have „inverted normals". Due to this mode the self-intersections of the surface are good apparent.

Now apply all the steps described above to the second dxf **3d** object, the „Steiner's Roman surface".

We have now two **3d** objects, two mathematical surfaces, which have in R^3 complicated structures with many self-intersections. The homeomorphism between them is not evident. Let's check it with help of the "Morph". The program "Morph" from the „Amorphium" Main menu supports up to three morph targets, using both vertex match and interpolated morphs. A continuous smooth deformation occurs only between homeomorphic objects which have the same number of vertices, number of triangles and mesh structure.
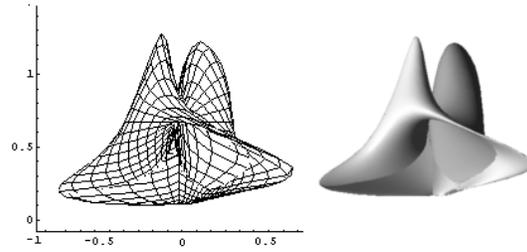
In the case of two constructed **3d** objects their mesh structures allow to make mutual morphing so that the topology of transforming objects does not change.

We can also propose a formula for the „between-surface" transformed from the Boy's surface at the moment t :

$$x_t\,(u,v) = t\,x_{Steiner}\,(u,v)\, +\, (1-t)\,x_{Boy}\,(u,v),$$

$$y_t\,(u,v) = t\,y_{Steiner}\,(u,v)\, +\, (1-t)\,y_{Boy}\,(u,v),$$

$$z_t\,(u,v) = t\,z_{Steiner}\,(u,v)\, +\, (1-t)\,z_{Boy}\,(u,v),$$
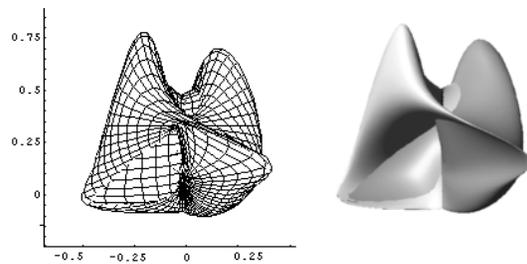
where t varies from 0 to 1; when $t = 0$, we have Boy's surface, when $t = 1$, we have Steiner's Roman surface.

Any sequence of ordered values t between 0 and 1 corresponds to a certain sequence of surfaces, which can be generated with „Mathematica" (see the left images in every pair below) and render further in „Amorphium" (see the right images below).

These surfaces can be used in „Amorphium" as key frames for animation.

The surfaces shown below correspond to the following sequence of values t :

$$t = 0,\ 1/3,\ 2/3,\ 5/6,\ 1$$
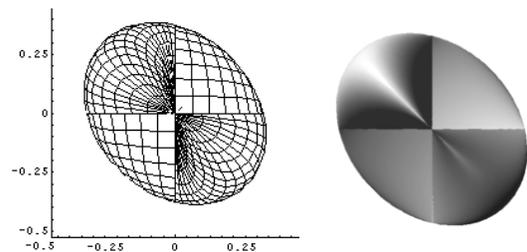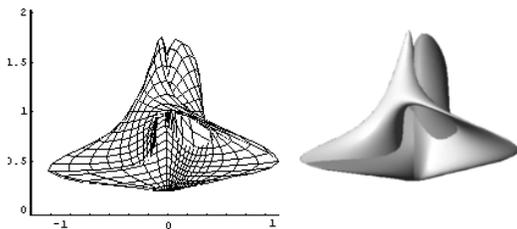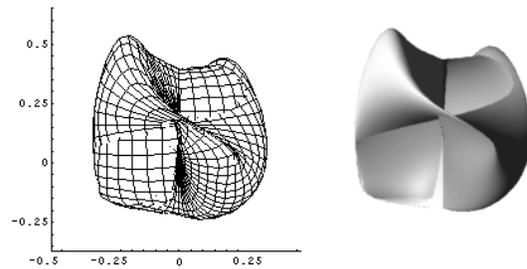
2. Eversion of a torus without a disk

The torus without a disk $T(r_1, r_2)$ having radius r_1 of the minimal paralle and radius r_2 of meridians is homeomorphic to the inverted torus without a disk $T(r_2, r_1)$ with radius r_2 of the minimal parallel and radius r_1 of meridians.

Let's consider a torus without a disk and an inverted torus without disk as two „Amorphium" objects.

The meshes of a torus with „correct normals" have a mirror structure relative to the torus with „inverted normals" , what does not allow to carry out a strongly homeomorphic transformation with the program „Morph" between two tori analoguously to the previous case described above.

To perform an interesting homeomorphism in this case, the „Morph" needs more than two objects which can be called „Torus before an eversion" and „Inverted Torus".

We have to create some additional objects to define for „Morph" intermediate steps of the whole transformation.

But at first we create a torus within the software „Rhinoceros" application (Robert McNeel and Associated, www.rhino3d.com), because with help of this program we can occurately cut a hole of needed size and form from the torus using a Boolean Subtract operation.

After that we export the obtained **3d** object „Torus without a disk" to „Amorphium" and construct the corresponding two-sided front-back coloured **3d** object by the procedure, described in part 1 of this paper.

The program "Morph" from the „Amorphium" Main menu can work with objects derived from the same original object; we can load in an object, make a copy of it, apply to it the „Amorphium" Tools, which do not change the topological properties of objects (such as „Stretch", „Bend", „Twist", „Taper", „Belly", „Brush", „Smooth") and then apply „Morph" between the original object and the copy.

In such a way we derive from the initial „Torus without a disk" some surfaces, which are intermediate steps in certain moments t of the torus eversion, they are similar to the „between-surfaces" from the part 1 of this paper. Then we use these surfaces as key animation frames.

That means, in the first case we have used surfaces which were constructed analytically (part 1 of the video), in the second case (part 2 of the video) we apply 3d modeling to the geometric objects created whithin the programs.

Conclusion

Many interesting three-dimensional mathematical objects are difficult to visualize. Even simple visualization should be sometimes helpfull to find new theorems as well as new proof ideas of known theorems. One of the approaches here is to combine mathematical software and software for

designers. „Amorphium" is a package targeted toward designers, but his features allow to use it with „Rhinoceros", „Mathematica" and other usual software for mathematicians and engineers.

3d objects having sufficiently complicate structures can be successfully imported to „Amorphium" from a variety of common 3d formats. We can render these objects as photorealistic one- or two-sided and work with them, for example, rotate and scale them dragging the mouse in the object's window; increase or decrease its polygon count; rotate light around an object to check details; find the areas with „inverted normals" of non-orientable surfaces; check their connectedness. We can make various topological operations with rendered objects simply dragging the mouse right or left; this possibility should be interesting for topology students and teachers, as well as scientists whose work can be facilitated by visualization.

References

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3. George K. Francis. *A Topological Picture book.* Springer, 1987