Optimal Deterministic Mixture Sampling

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Abstract

Multiple Importance Sampling (MIS) can combine several sampling techniques preserving their advantages. For example, we can consider different Monte Carlo rendering methods generating light path samples proportionally only to certain factors of the integrand. MIS then becomes equivalent to the application of the mixture of individual sampling densities, thus can simultaneously mimic the densities of all considered techniques. The weights of the mixture sampling depends on how many samples are generated with each particular method. This paper examines the optimal determination of this parameter. The proposed method is demonstrated with the combination of BRDF sampling and Light source sampling, and we show that it not only outperforms the application of the two individual methods, but is superior to other recent combination strategies and is close to the theoretical optimum.

Keywords: global illumination, rendering equation analysis, multiple importance sampling, Monte Carlo.

1. Introduction

Multiple Importance Sampling (MIS) [VG95, Vea97] has been used and proven efficient in many Monte Carlo rendering algorithms. It is able to preserve the advantages of the combined techniques and requires only the calculation of the probability density (pdf) of all methods when a sample is generated with one particular method. The weighting scheme applied in MIS depends on the pdfs of the individual techniques and also on the number of samples generated with each of them. Traditionally, these sample numbers are equal [Vea97] or are set with some manual experimentation.

This paper proposes a simple adaptive technique to automatically determine the sampling budget based on the statistics of previous samples. Adaptively determining the sampling strategy may be dangerous because it can result in a biased sampling method. However, in our case, MIS offers unbiasedness if the domain of nonzero integrand is sampled with positive pdf and only the weighting scheme is modified.

In this paper we present two variants of adaptive MIS sampling scheme for combining two strategies and demonstrate advantages of the new method with the classical scene of Veach [VG95].

2. Previous work

MIS has been applied in a huge number of rendering algorithms, but their review is beyond the scope of this paper. We concentrate on papers aiming at the allocation of the sample budget. It has been proven that it is worth deciding on the number of samples deterministically rather than randomly selecting from the techniques using the weight as the selection probability [SHSK16]. Several estimators have been proposed that are better than balance heuristics with the equal sample budget [SHSK16, HS14, SH17], and it has been proven [SHSK18] that the improvement can be larger than suggested by [Vea97]. Sbert et al. [SHSKE18] proposed to make the weight inversely proportional to the variance and the second moment, respectively, and also considered the cost associated with the sampling strategies. Lu et al. [LPG13] proposed an adaptive sampling algorithm for environment map illumination. Adapting between BRDF sampling and environment map sampling, they used the Taylor series approximation of the variance around equal sampling. In [EMLB15a, EMLB15b] strategies and analysis were given assuming equal number of samples.

3. The proposed method

Suppose we wish to estimate the value of integral $I = \int f(x)\mathrm{d}x$ and we have two proposal pdfs $p_1(x)$ and $p_2(x)$ to generate random samples in the domain of the integral. A practical example is the direct lighting problem where we can generate random rays with *light source sampling* and *BRDF sampling*. If the total sampling budget is N, we should decide on the number N_1 of samples generated with $p_1(x)$, which let the other sampling technique generate $N - N_1$ samples with $p_2(x)$. Let us denote the fraction of samples generated by the first method by $\alpha = N_1/N$. The selection of this ratio affects the variance of the estimator, and our goal is to set it quasi-optimally.

DOI: 10.2312/egs.20191018

MIS with balance heuristics [VG95] is equivalent to assuming that the pdf is the random mixture of the two original proposal pdfs:

$$p(x) = \alpha p_1(x) + (1-\alpha) p_2(x). \tag{1}$$

The primary MIS estimator is:

$$F = \frac{f(X)}{p(X)} = \frac{f(X)}{\alpha p_1(X) + (1-\alpha) p_2(X)}. \tag{2}$$

The expected value of this estimator is

$$\mu = E_p[F] = \int f(x)\mathrm{d}x,$$

i.e., it is an unbiased estimator independently of α if any x of nonzero integrand is generated with positive probability density, where $E_p[F]$ expresses that samples X should be generated with pdf $p(x)$.

Unlike the expected mean value, its variance does depend on α. We use the variance of the random mixture to approximate the smaller variance of the deterministic mixture sampling:

$$V_p[F](\alpha) = \int \frac{f^2(x)}{\alpha p_1(x) + (1-\alpha) p_2(x)}\mathrm{d}x - \mu^2.$$

To get a low variance estimator, the variance is minimized by controlling weight α. The variance is minimal if the first derivative $V'_p[F](\alpha)$ is zero and the second derivative $V''_p[F](\alpha)$ is positive:

$$V'_p[F](\alpha) = \frac{\partial V_p[F](\alpha)}{\partial \alpha} = \int \frac{f^2(x)(p_2(x) - p_1(x))}{p^2(x)}\mathrm{d}x = 0, \tag{3}$$

$$V''_p[F](\alpha) = \frac{\partial^2 V_p[F](\alpha)}{\partial^2 \alpha} = 2\int \frac{f^2(x)(p_2(x) - p_1(x))^2}{p^3(x)}\mathrm{d}x \geq 0. \tag{4}$$

Note that the second condition always holds if α is in $[0,1]$, i.e. here the variance is a strictly convex function of α. If the first derivative is zero, then it is global optimum [SH17].

Fast numerical solution can be obtained with the Newton-Raphson method. Let us approximate the derivative of the variance $V'_p[F](\alpha)$ with the linear term of the Taylor's series around the current estimate α_n in iteration n:

$$V'_p[F](\alpha) \approx V'_p[F](\alpha_n) + V''_p[F](\alpha_n)(\alpha - \alpha_n). \tag{5}$$

From this, the new α_{n+1} should make the approximated derivative equal to zero, thus we can compute it as

$$\alpha_{n+1} \approx \alpha_n - \frac{V'_p[F](\alpha_n)}{V''_p[F](\alpha_n)}.$$

Weight α should be in the unit interval. Moreover, we approximate the variance derivatives with finite number of samples, which may require that both methods are sampled for sure. Therefore, these issues are solved by limiting the α range to $[0.1, 0.9]$, and whenever Newton-Raphson iteration presents a value outside of it, we replace it with the limit. Based on these estimates, we can adapt the weight during the sampling process. This method is called the *Root Adaptive MIS*. The condition for the first derivative to be zero can be written into two equivalent forms of expected values. These options are considered separately, as these expectations are estimated differently from the already taken samples.

3.1. Derivative of the variance: Version 1

From Eq. 3 the Version 1 formula is written as

$$V'_p[F](\alpha) = E_{p_2}\left[\frac{f^2(x)}{p^2(x)}\right] - E_{p_1}\left[\frac{f^2(x)}{p^2(x)}\right] \tag{6}$$

where $E_{p_i}\left[f^2(x)/p^2(x)\right]$ is the expectation of $f^2(x)/p^2(x)$ when samples are generated by Method i, i.e. when only Method i is applied:

$$E_{p_i}\left[\frac{f^2(x)}{p^2(x)}\right] \approx \frac{1}{N_i}\sum_{j=1}^{N_i} \frac{f^2(X_{i,j})}{\left(\alpha p_1(X_{i,j}) + (1-\alpha) p_2(X_{i,j})\right)^2}. \tag{7}$$

3.2. Derivative of the variance: Version 2

Eq. 3 can also be transformed into our Version 2 formula:

$$V'_p[F](\alpha) = E_p\left[\frac{f^2(x)(p_2(x) - p_1(x))}{p^3(x)}\right] \tag{8}$$

where $E_p\left[f^2(x)(p_2(x) - p_1(x))/p^3(x)\right]$ is the expectation of $V'_p[F](\alpha)$ when samples are taken by mixture sampling from $p(x)$.

This formula leads to the following estimator of the derivative:

$$V'_p[F](\alpha) \approx \frac{1}{N}\sum_{i=1}^{2}\sum_{j=1}^{N_i} \frac{f^2(X_{i,j})(p_2(X_{i,j}) - p_1(X_{i,j}))}{\left(\alpha p_1(X_{i,j}) + (1-\alpha) p_2(X_{i,j})\right)^3}. \tag{9}$$

3.3. Second derivative of the variance

The second derivative of the variance contains a higher degree polynomial of the individual densities, so it does not simplify to simple forms as in the case of the first derivative. So for the estimation of the second derivatives, we consider only Version 2, which uses MIS also for its estimation:

$$V''_p[F] = 2E_p\left[\frac{f^2(x)(p_2(x) - p_1(x))^2}{p^4(x)}\right] \approx \frac{1}{N}\sum_{i=1}^{2}\sum_{j=1}^{N_i} \frac{f^2(X_{i,j})(p_2(X_{i,j}) - p_1(X_{i,j}))^2}{\left(\alpha p_1(X_{i,j}) + (1-\alpha) p_2(X_{i,j})\right)^4}. \tag{10}$$

3.4. Robust computation of variance derivatives from discrete samples

The question is whether Eq. 6 or Eq. 8 can be estimated with lower variance. The answer depends on the variance of the primary estimators and the number of samples. Note that in the first method, the two expectations are estimated with the number of samples of the individual methods, respectively, while the second approach uses the total number of samples. Thus, from this point of view, the second method offers lower variance. However, the primary estimator may lead to different preference. Assume that the MIS estimator is good enough, i.e. $f(x)/p(x)$ is approximately constant. In this case, the primary estimator for Eq. 6 is close to zero, but it is not for Eq. 8.

The control of weight α requires the approximation of the variance derivatives from discrete samples as described by Eqs. 7, 9 and

10. The approximation error should be small. Unfortunately, the discrete samples depend on the current weight α, which changes in every iteration. Keeping every single sample separately to allow the update when α changes is not an option due to the very high number of samples. Using only the samples of the current iteration, on the other hand, does not exploit all samples and ignores relevant information, leading to numerically unstable weights. The solution is to reuse aggregated samples from previous iterations as well, having approximately modified them to follow the change of α. When α is updated, we use Eq. 5 to find an approximate new value of the first derivative. We could use a similar technique to update the second derivative with the third, but our current implementation just keeps the second derivative estimate as it is. Note that such an update is only an approximation, thus the update value of the previous iteration is taken into account with less weight as the discrete values of the current iteration. Note that if all contributions were given the same importance, a sample in iteration step n would get $1/n$ weight while all previous samples would be multiplied by $1-1/n$. To emphasize later samples, the current samples get weight $1/\sqrt{n}$ and aggregated estimate from the previous iteration gets weight $1-1/\sqrt{n}$.

4. Results

In order to test the proposed method, we take the classic scene of Veach with combined light source and BRDF sampling. The four shiny rectangles have physically plausible Phong-like BRDF with shininess parameters 500, 1000, 5000, and 10000, respectively. The four spherical light sources emit the same power.

We allocated 1000 samples per pixel organized in 20 iterations. In the first iteration 25 BRDF and 25 light source samples per pixel were used to estimate the variance, second moment and the derivatives for Lu's method, and the other iterations utilized the per-pixel α weights computed from these values to find the optimal number of light source and BRDF samples. Fig. 1 shows the rendered images together with the obtained α maps.

Fig. 2 depicts the RMSE values after each iteration when the α weights are refined at the end of each iteration. Note that equal sample count MIS is better than both light source sampling and BRDF sampling, but using sophisticated methods to find the weights α further improves MIS. In addition to the original sampling techniques and equal count MIS, we also compared the proposed method to the inverse second moment approach of Sbert et al. [SHSKE18] and Lu's method [LPG13].

The relative efficiency values with respect to equal count MIS are shown by Table 1. We can observe that the new methods offer 30-40% improvement in efficiency. We also examined the theoretical optimum, i.e. how much improvement can be obtained at all with the proper definition of the weight. To do so, we computed the optimal α values in each pixel with a global optimization algorithm before starting the rendering phase, and kept using these values during sampling. Note that this is not a practical approach, but shows how far on-the-fly techniques are from the theoretical optimum.

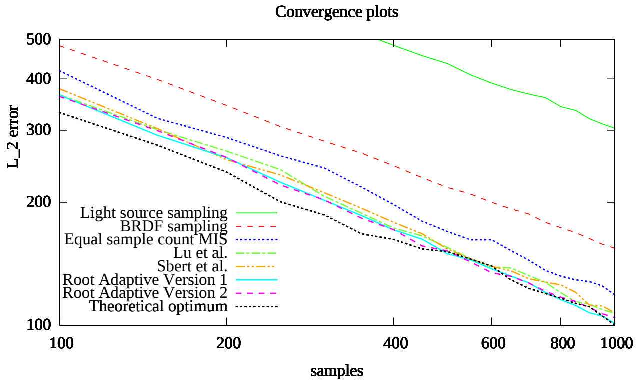

Figure 2: *RMSE plots as functions of the number of samples per pixel.*

Method	Relative efficiency
BRDF sampling	0.59
Light source sampling	0.16
Equal sample MIS	1
Lu et al. [LPG13]	1.24
Sbert et al. [SHSKE18]	1.24
Root adaptive. Version 1	1.40
Root adaptive. Version 2	1.31
Theoretical optimum	1.43

Table 1: *Relative efficiency with respect to equal sample count MIS.*

5. Conclusions

In this paper we investigated the problem of determining the sample budget for techniques combined by Multiple Importance Sampling. The proposed method calculates the weighting factors of two combined techniques iteratively in parallel with sample generation using more and more stable statistics. We have shown that such adaptation is worth doing since we can gain 40% in efficiency with respect to equal count sample distribution for the classical scene of Veach. Note that with this number, we are close to the 43% theoretical optimum. In future, the technique could be extended for different sampling costs to optimize for sampling efficiency.

Acknowledgements

Mateu Sbert acknowledges the support of grant TIN2016-75866-C3-3-R from Spanish Government, Vlastimil Havran acknowledges the support of the OP VVV MEYS funded project CZ.02.1.01/0.0/0.0/16_019/0000765 Research Center for Informatics, and László Szirmay-Kalos has been supported by OTKA K–124124, Hungary.

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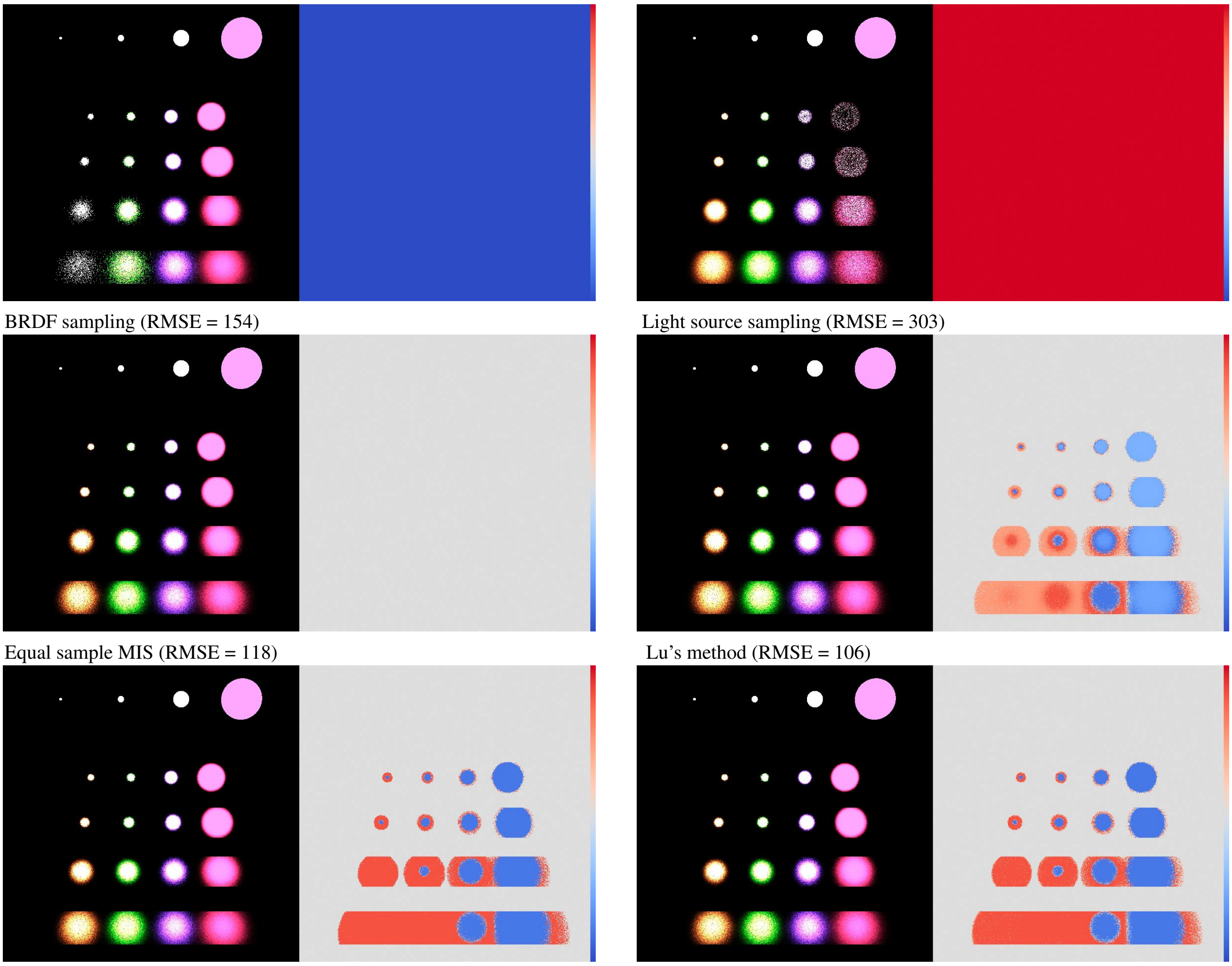

Figure 1: *Comparison of MIS weighting schemes. The left part is the rendered image, the right part is weight* α *of the light source sampling. The images were saved after tracing 100 rays per pixel.*

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