Extraction of Critical Points and Nets Based on Discrete Gradient Vector Field

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Abstract

In this paper we address the problem of representing the topology of discrete scalar fields defined on triangulated domains. To this aim, we use the notion of discrete gradient vector field that we have introduced in [4] to classify the critical points of a scalar field defined over a two-dimensional domain that generalizes the nature of critical points in the differentiable case. Then we sketch an algorithm for extracting a critical net in the discrete case.

1 Introduction

Morse theory has been used in geometric modeling to extract features from a scalar field defined at the vertices of a triangulated manifold domain [1, 2, 3, 5]. The scalar field is usually described by a (at least) C^2-differentiable Morse function [10], i.e., a function in which the critical points are non-degenerate. Thom [14] followed by Smale [13] have shown that, when f is a Morse function, the domain can be decomposed into cells, each of which corresponds to a critical point and has a dimension equal to the index of the critical point. A *CW*-complex representation of the domain can be, therefore, given (see also [8]). Smale has shown that critical points are connected together in a certain order on their indexes by some integral curves called *separatrices*. A *critical net*, also called a *critical graph* [9, 11], can be built to represent the topology of the field. Each node of the graph represents a critical point, while each edge corresponds to a separatrix connecting two nodes. This representation can help, for instance, to understand the topology of the level sets (isolines, isosurfaces, or implicit surfaces) propagating along the net, (see [7]).

There have been some contributions [1, 2, 5] that use approximations of the scalar field by interpolating the values of the field at the vertices of the mesh by piecewise linear functions, to get some similar geometric properties as Morse functions. In [5] differentiability simulation has been introduced to build quasi-Morse complexes for piecewise linear functions on compact 2-manifolds without boundary. Hierarchical coarser complexes were built based on a persistence notion for critical points.

Recently, Forman introduced [6] a discrete definition of Morse functions for cell complexes. He has proved analogous results as in the differentiable Morse theory. Forman functions are defined on all cells of the complex, while, in practice, scalar fields are only known at the vertices (i.e., 0-cells) of a simplicial mesh. A scalar field can be easily extended to a (non-unique) Forman function defined on all cells, but in this case the simplification process proposed by Forman stops after few steps and all remaining cells in the complex become critical. This fact prevent us from applying Forman theory to build efficient multiresolution models based on Forman functions.

In [4] we have proposed a modification of the notion of discrete gradient vector field, introduced by Forman, to handle discrete scalar fields, which has led us to define what we called a Smale-like

decomposition for scalar fields in arbitrary dimensions.

Here, we introduce a classification of critical points in the discrete case, which generalizes the classical nature of critical points for C^n-differentiable functions with $n \geq 2$. The differentiability simulation that we obtain corresponds to a C^1-piecewise differentiable function. In [12, 15], a local study of critical points is performed for gray-scale images, which classifies them into pits, peaks and passes, but which does not go beyond a generalization of such classical notions.

Combining our classification of critical points with the Smale-like decomposition, we can construct a critical net connecting different critical points. This critical net generalizes the classical critical net since classical edges become, in general, thick bands pinched at their extremes (which are the generalized critical points).

The remaining of this paper is organized as follows. In Section 2, we briefly review the basic notions from Morse theory and combinatorics that we need for the remaining material. In Section 3, we describe our Smale-like decomposition algorithm (which works for fields in any arbitrary dimension) with some of its properties, and we construct the discrete gradient vector field. In Section 4, we introduce a classification of critical points in the two-dimensional case. Finally, in Section 5, we briefly sketch a technique to extract a critical net in two and three dimension and illustrate it by an example.

2 Basic notions

In this Section, we summarize some results from Morse theory necessary to understand the topological structure of the domain associated with a scalar field. We then recall some combinatorial notions that we use throughout this paper. A *Morse function* on R^d is a differentiable (at least C^2) real-valued function f defined on R^d such that its *critical* points are isolated [10]. This means that the Hessian matrix $Hess_P f$ of the second derivatives of f at a point $P \in R^d$ such that $Grad_P f = 0$ is non-degenerate. Since these properties are local, a Morse function can be defined on a manifold M (that is a topological space where the neighborhood of any point is homeomorphic to an open ball or to half of an open ball). Morse [10] has proved that there exists a local coordinate system $(y^1, ..., y^n)$ in a neighborhood U of any critical point P, with $y^j(P) = 0$, for all $j = 1, \ldots, n$, such that the identity

$$f = f(P) - (y^1)^2 - ... - (y^\imath)^2 + (y^{\imath+1})^2 + ... + (y^n)^2$$

holds on U, where $\imath$ is the number of negative eigenvalues of $Hess_P f$, and it is called the *index* of f at P.

The above formula implies that the critical points of a Morse function are isolated. This allows us to study the behavior of f around them, and to classify their nature according to the signs of the eigenvalues of the Hessian matrix of f. If the eigenvalues are all positives, then the point P is a *strict local minimum* (a pit). If the eigenvalues are all negatives, then P is a *strict local maximum* (a peak). If the index $\imath$ of f at point P is different from 0 and n, then the point P is neither a minimum nor a maximum, and, thus, it is called an *$\imath$-saddle* point (a pass).

The decomposition of the manifold domain, on which function f is defined, introduced by Thom [14] and followed by Smale [13] is based on the study of the growth of f along its integral curves. An *integral curve* is a curve everywhere tangent to the gradient vector field $Grad f$. The classical Taylor formula shows that integral curves follow the (gradient) directions in which the function has the maximum increasing growth. Hence, integral curves cannot be closed, nor infinite (in a compact manifold), and they do not intersect each other. They are emanating from critical points, or from boundary components of the domain and reach other critical points, or boundary components. The integral curves originating from a critical point of index $\imath$ form a $\imath$-cell C^s, called a *stable manifold*. In the same way integral curves converging to a critical point of index $\imath$ form a dual $(n - \imath)$-cell C^u, called an *unstable manifold*. Because of the properties of the integral curves, the stable manifolds are pair-wise disjoint and decompose the field domain M into open cells, see Figure 1(a). The cells form a complex, as the boundary of every stable manifold is the union of

lower dimensional cells. Similarly the unstable manifolds decompose M into a complex dual to the complex of stable manifolds.

Figure 1(a) gives an example of a stable decomposition of a two-dimensional domain. The function is assumed to be a Morse function. It has three minima (shown by •), two maxima (shown by ⊙), and five saddle points (shown by ⋈). Integral curves originate from each minimum in all directions and from the right side of the boundary. Each integral curve converges either to a saddle, a maximum or to a boundary component. Two integral curves originate from each saddle point. Integral curves originating from a minimum (or from the right side boundary) sweep a 2D cell, while integral curves emanating from a saddle form a segment containing the saddle point in its interior. Integral curves connecting saddles to critical points are called *separatrices*. The graph formed by all separatrices with their end points is called a *critical net*. Figures 1(b) illustrate a critical net corresponding to stable decomposition in Figure 1(a).
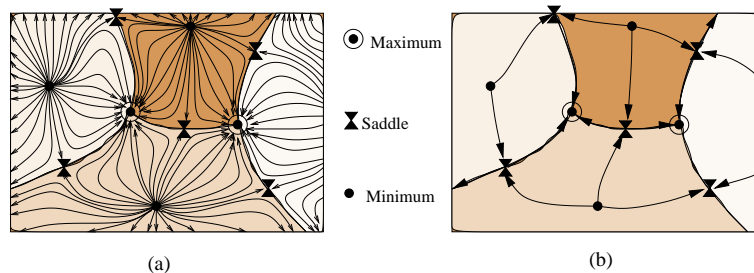

Figure 1: (a): Decomposition of a domain into four stable 2-manifolds; (b) Critical net corresponding to decomposition in (a).

Let now k be a non-negative integer, a *k-simplex* or a *k- dimensional simplex* is the convex hull of $(k+1)$ affinely independent points in R^d (with $k \leq d$) called *vertices*. A *face* σ of a k-simplex γ, $\sigma \subseteq \gamma$, is a j-simplex $(0 \leq j \leq k)$ generated by $(j+1)$ vertices of γ. A *simplicial complex K* is a collection of simplexes, also called *cells*, such that if γ is a simplex in K, then each face $\sigma \subseteq \gamma$ is in K, and the intersection of two simplexes is either empty or a common face. The *dimension* of a simplicial complex K is the maximum among the dimensions of its simplexes. A *maximal* simplex in K is any simplex which has the same dimension as K. A *top* simplex in K is a simplex which is not the proper face of any other simplex in K. The *carrier* $| K |$ of a simplicial complex K is the space of all points in simplexes of K. In this case, K is called a *triangulation* of $| K |$. A manifold endowed with a triangulation is called a *combinatorial manifold*.

Let K be a simplicial complex and γ be a cell in K. The *star* of γ is the set $St(\gamma)$ of all cells in K which are incident at γ. Thus, $St(\gamma) = \{\sigma \in K : \gamma \subseteq \sigma\}$. The star of γ describes the neighborhood of γ in the complex. The closure of a set of cells Γ is the smallest subcomplex $\overline{\Gamma}$ of K containing Γ. Clearly, $\overline{\Gamma}$ consists of all cells of Γ plus their faces.

The *link* of cell γ is the sub-complex $Lk(\gamma)$ of K defined as $Lk(\gamma) = \overline{St}(\gamma) - St(\overline{\gamma})$, where $\overline{\gamma}$ is the closure of γ. The link describes the boundary of $St(\gamma)$. A *cone* from a vertex w to a simplex γ is the convex combination of all vertices of γ with w. We denote it by (γ, w). If w is affinely independent of the vertices of γ, then the cone from w to γ is a simplex of dimension $dim(\gamma) + 1$, where $dim(\gamma)$ denotes the dimension of γ.

3 Smale-like Decomposition for a discrete scalar field

In [4] we have introduced the notion of *discrete gradient vector field "Gradf"* associated with f. The *negative* discrete gradient vector indicates the directions in which the scalar field is decreasing. It starts from local maxima and ends at other critical points. The negative gradient vector field originating from a local maximum characterizes a component, called an *unstable Smale complex*, in the domain M on which the scalar field is defined. Unstable Smale complexes subdivide the domain M into an assembly of complexes each of which corresponds to a local maximum and that

we call a *Smale-like decomposition* of the domain M. In this section, we give an overview of this decomposition and of the construction of the negative gradient vector field. Note that in the same way, the (*positive*) gradient vector field originating from local minima can be defined and its stable complexes built.

Let us consider a triangulated domain M (which is not necessary a manifold) and a scalar field f defined at the vertices of the triangulation of M. Let K be the simplicial complex associated with the triangulation of M. We assume that $f(u) \neq f(v)$ for all vertices $u \neq v$ (this can be obtained by a small perturbation of the scalar field f). We keep a current complex K' which is initialized to be equal to K. We consider a vertex v in K' corresponding to the global maximum of the scalar field f. The values of f at the vertices of $St(v)$ are, thus, less than $f(v)$. We define the component $C(v)$ corresponding to v to be the same as $\overline{St(v)}$, and we initialize the boundary $\partial C(v)$ of $C(v)$ as $Lk(v)$. Then, for each top simplex γ in $\partial C(v)$ that is incident to another simplex (γ, w) in $K' - C(v)$, we compare the values of f at vertices of γ with $f(w)$. If $f(w)$ is less than all of them, we extend $C(v)$ to be $C(v) \cup \overline{(\gamma, w)}$ and we replace γ in $\partial C(v)$ with the faces of cone (γ, w) that contain w. We continue extending $C(v)$ in such a way that $C(v)$ is a region on which f decreases. The process stops when $C(v)$ cannot be extended any further while maintaining the above property. Then $C(v)$ is deleted from K', and we repeat the process.

The result is, thus, a decomposition $\mathcal{D}$ of K into *unstable Smale complexes* $C_i \equiv C(v_i)$, each of which corresponds to a maximum of the scalar field f. We note that in some cases where $v \in \partial C(v)$ point v can be canceled and therefore component $C(v)$ is merged with another component. We will discuss this situation below.
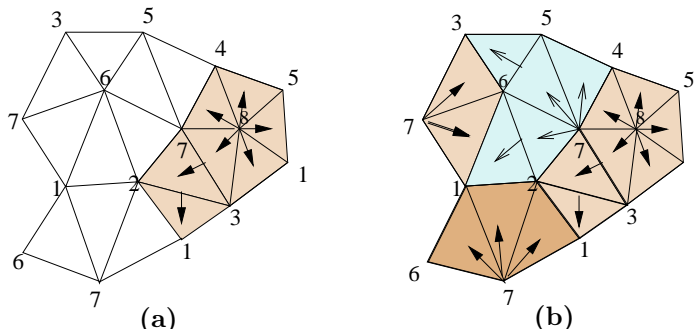

Figure 2: In (a), the decomposition process of K: component corresponding to the point with $f = 8$ with its discrete gradient vector field. In (b), the final components decomposing K and their discrete gradient vector field are shown.

Figure 2(a) shows the process of growing a component, while Figure 2(b) we show the final decomposition of the complex in (a) with its gradient vector field. Each shaded region corresponds to an unstable Smale component. In Figure 3, we show the results produced by the decomposition algorithms on a terrain data set representing Mount Marcy (courtesy of USGS), which contains 17334 triangles.

The decomposition algorithm described above allows us to define a discrete form of the gradient vector field for a scalar field f. A *discrete (negative) gradient vector field* is defined by two functions: (a) a multi-valued function ϕ associates each local maximum v, which corresponds to a component $C(v)$ of K, with the top cells γ' in $St(v)$, i.e., $\phi(v) = \{\gamma' : \gamma'$ is a top cell in $St(v)\}$; (b) a multi-valued function ψ associates with each cell γ in $C(v) - \{\partial C(v) \cup St(v)\}$, which has been used in the extension process, the cones (γ, w_i) added to γ. Equivalently, vertices (w_i) are sufficient to characterize function ψ).

Functions ϕ and ψ define the discrete (negative) gradient vector field of f. As in the differentiable case, the gradient field denotes the directions in which the function decreases, and characterizes the critical cells and points. Critical cells are those on which ψ is not defined. To illustrate functions ϕ and ψ, we draw vectors, from the initial vertex v, to all top cells in $St(v)$, and
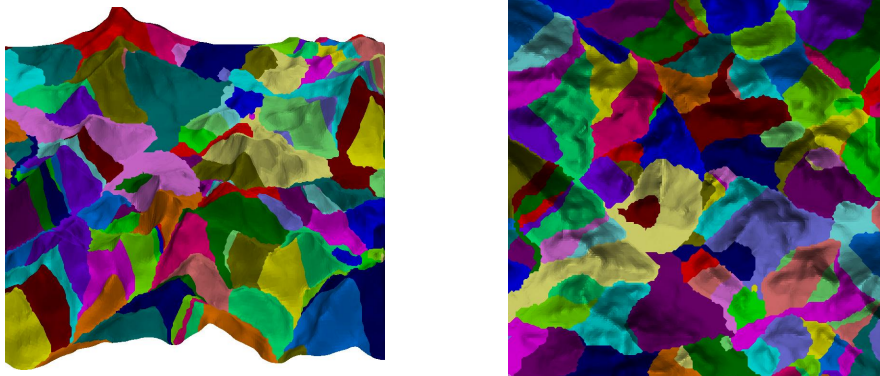
Figure 3: The image corresponds to Mount Marcy (courtesy of USGS) with 17344 triangles. The algorithm decomposes the triangle mesh its into 119 components.

a vector from each cell γ to the cones (γ, w_i) used in the decomposition process as shown in Figure 2. Vectors are originating, from a local maximum, in all possible directions. A local minimum receive vectors from all possible directions, while a saddle point has vectors originating and entering in alternating groups of successive directions. When there are only two groups, the point will be called an *inflection* point. We call such points *critical points*. In a generic (non-manifold) complex, other cells should be defined as critical cells for the scalar field. This is due to the non-manifoldness of the complex.

4 Classification of Critical Points in the 2D case

In this Section we provide a classification of critical points in the 2D case based on the decomposition described in the previous section. Several cases are discussed and a differentiability simulation is introduced to understand the nature of each case and make easier the classification of critical points. Our classification generalizes the classical nature of critical points of Morse functions, used in geometric modeling [5, 7, 8, 9, 11] to extract significant features of scalar fields. Our differentiability simulation corresponds to a piecewise C^1-differentiable function.

4.1 Classification with respect to a single component

Let K be a triangle mesh endowed with a scalar field f defined at the vertices of mesh. Let $C(v)$ be a component of the Smale-like decomposition associated with the local maximum v. Since vertex v has been chosen to be a maximum in K, then v is a critical point. The gradient vector field is emanating from v to-wards all triangles in the star of v. The local behavior of the scalar field around a point w on the boundary $\partial C(v)$ give rises to several possible cases. We illustrate some of them in Figures 4 and 5. All possible cases can be treated in a similar way. Critical points are classified with respect to each component to which they belong as maxima, minima, saddle and slope points. We illustrate a simulation of the differential case in each case by showing the local behavior of the gradient vector field in the differentiable case.

Differentiability simulation for a maximum is illustrated in Figure 6, for a saddle point in Figure 7, for a minimum is illustrated in Figure 8(a), and for a slope in Figure 8(b).

4.2 Local Classification of Critical Points

The previous classification is relative to the component in which we consider the critical point. But, in general, a critical points belongs to several components. By re-grouping all situations relative to components incident at a critical point w, we can define the nature of w with respect
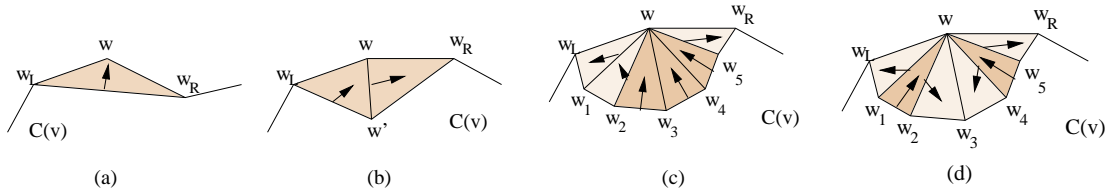
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Figure 4: Local behavior of the scalar field around a point w on $\partial C(v)$: (a) corresponds to a minimum; (b) corresponds to a slope; (c) and (d) correspond to a general saddle situation.
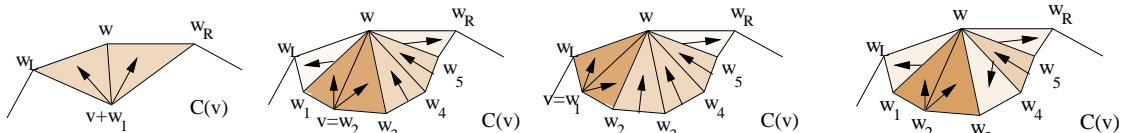

Figure 5: General saddle situations in the special case in which v belongs to the star of w

to the whole domain. In Figures 9-11, we illustrate the cases in which only two components are incident at a critical point. In Figures 12-14, we illustrate the general case of an arbitrary number of components incident at the same critical point. Thus, we can classify the point as a minimum, an inflection point, a monkey saddle, or as a generalized saddle point. We talk about "generalized" point since in several cases, the gradient vector field converges, or diverges, from the saddle point in more than four separatrices and which are generally thick bands pinched at extremities (we call them separatrix bundles). Maxima correspond to the inner points v of components $C(v)$. Note that, in the case of an inflection point, the differentiability simulation allows us to eliminate the critical point by smoothing a long, thin neighborhood of the boundary, as shown in Figure 10(a). Moreover, the canceling operation an inflection point v which is a maximum for a component $C(v)$, allows us to merge this component with other components incident to the same point v, see Figure 10(b).

When the critical points for a scalar field are maxima, minima or saddle points (with 4 separatrices), we say that the scalar field defines a *discrete Morse function*. The properties of such discrete Morse functions, their interaction with general scalar fields, and the properties of their gradient vector fields will be studied in our future work.

5   Extraction of Critical Nets

In this Section, we sketch our on-going work on extraction of critical nets by using the characterization of critical points and the decomposition algorithm described in the previous sections. This allows us to extract critical nets in both two-dimensional and three-dimensional cases.

As we have seen from differentiability simulation in Section 4, the negative gradient vector field converges to a minimum m from all directions in its neighborhood. Around a saddle point w, there exist alternative groups of separatrix bundles in which the negative gradient vector field converges to (or diverges from) w, as shown in Figures 7. For C^n-differentiable functions each group reduces to just one separatrix leaf. The union of such separatrix bundles or leaves forms the *critical net*. A critical net can be easily extracted from our decomposition algorithm by performing two decompositions to find stable and unstable components. The boundaries of such dual components intersect transversally.

In the two-dimensional case, the union of these boundaries forms a graph that connects maxima to saddles, and saddles to minima. The critical net is, then, defined to be the union of these boundaries.

In the three-dimensional case, the boundaries of two dual components are surfaces that intersect in a curve which contains 2- and 1-saddles points. Moreover, we note that 2-saddles are characterized discretely as local maxima in $\partial C(v)$. The two-dimensional complex $\partial C(v)$ must be decomposed by our 2D-algorithm to extract its 1-saddles and minima and connect them in a (par-
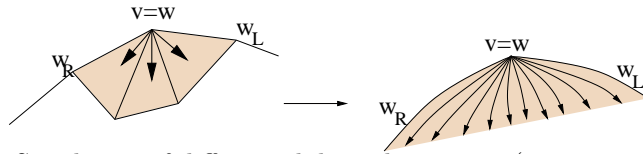
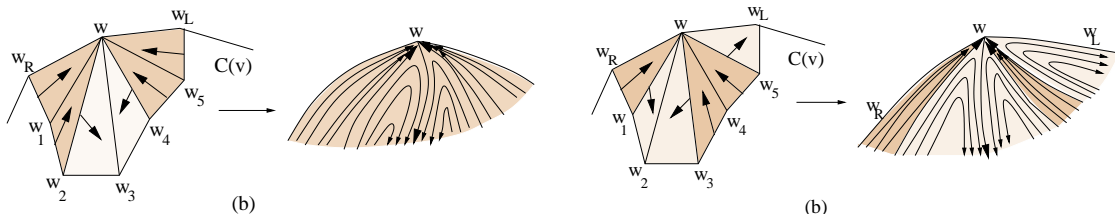Figure 6: Simulation of differentiability when $v = w$ (i.e, w is a maximum)

Figure 7: In both (a), (b), the gradient vector field converges to w and diverges from w in alternating directions; w is a saddle point.

tial) critical graph. These saddles in $\partial C(v)$, are the 1-saddles for the component $C(v)$. The same process should be done for the dual components of $C(v)$ to extract the curves that connect the maximum v to 2-saddles on $\partial C(v)$. The final result, containing all partial graphs and separatrices connecting maxima to 2-saddles, defines the critical net.

Figure 15(a) gives both stable and unstable decomposition of the scalar field already represented in Figure 3. The corresponding critical net is depicted in Figure 15(b).

6   Conclusion and Future Work

In this work, we have reviewed an algorithm that decomposes the domain of a d-dimensional scalar field into regions similar to Morse manifolds [4]. We have described a classification of critical points that generalizes the nature of critical points in the differentiable case and we have sketched an algorithm for extracting critical nets for given scalar fields. Example of such notions have been presented, and a natural definition of a discrete Morse function has been proposed.

On-going work experiments the algorithms for classifying critical points in 2D and for extracting critical nets with bundles. We are also working on a classification of critical points in the 3D case. In the future, we plan to extend our study to non-manifold simplicial complexes in 2D and 3D cases, and investigate a hierarchical representation for a Smale-like decomposition as the basis for a compact representation for multiresolution scalar fields in the discrete case.

Acknowledgments

References

[1] C. L. Bajaj, V. Pascucci, and D. R. Shikore. Visualization of scalar topology for structural enhacement. In *Proceedings of the IEEE Conference on Visualization '98 1998*, pages 51–58, 1998.
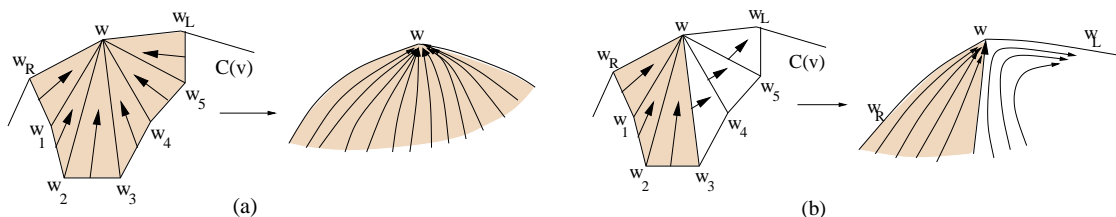
Figure 8: In (a) the gradient vector field converges to w from all directions around w which is a minimum. In (b) the gradient vector field converges to w on the region located on the right of w, and diverges from w on the region on the left. Point w is a slope point.
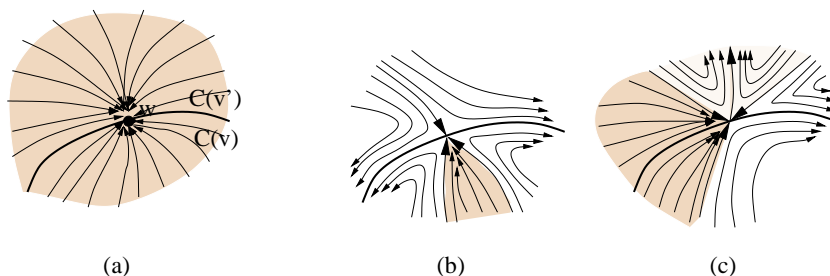

Figure 9: Nature of a critical point common to *two* components. In (a), point w is a minimum in both $C(v)$ and $C(v')$, then it is a minimum in K. In (b), point w is a saddle point in both $C(v)$ and $C(v')$, then it is a saddle in K. In (c) point w is a slope in both $C(v)$ and $C(v')$, then it becomes an inflection point in K.

[2] C. L. Bajaj and D. R. Shikore. Topology preserving data simplification with error bounds. *Journal on Computers and Graphics*, 22(1):3–12, 1998.

[3] S. Biasotti, B. Falcidieno, and M. Spagnuolo. Extended Reeb graphs for surface understanding and description. In *Proc. 9th DGCI'2000, Upsala*, volume LNCS 1953, Springer-Verlag, pages 185–197, 2000.

[4] L. De Floriani, M.M. Mesmoudi, and E. Danovaro. A smale-like decomposition for discrete scalar fields. In *Proceedings International conference on pattern recognition (in print)*, Quebec City, August 2002.

[5] H. Edelsbrunner, J. Harer, and A. Zomorodian. Hierarchical morse complexes for piecewise linear 2-manifolds. In *Proc 17th Sympos. Comput. Geom.*, pages 70–79, 2001.

[6] R. Forman. Morse theory for cell complexes. *Advances in Mathematics*, 134:90–145, 1998.

[7] J.C Hart. Morse theory for implicit surface modeling. In *H.C. Hedge and K. Poltihier, eds, Mathematical Visualization, Springer-Verlag*, pages 256–268, 1998.

[8] J.C Hart. Using the CW-complex to represent topological structure of implicit surfaces and solids. In *Proc. Implicit Surfaces 1999, Eurographics/SIGGRAPH*, pages 107–112, 1999.

[9] C Johnson, M. Burnett, and W. Dumbar. Crystallographic topology and its applications. In *Crystallographic Computing 7: Macromolecular Crystallography data, P. E. Bourne, K. D. Watenpaugh, eds., IUCr Crystallographic Symposia, Oxfor University Press,*, 2001.

[10] J. Milnor. *Morse Theory*. Princeton University Press, 1963.

[11] L. R. Nackman. Two-dimensional critical point configuration graph. *IEEE Transactions on Pattern Analysisand Machine Intelligence*, PAMI-6(4):442–450, 1984.

[12] T. K. Peucker and E. G. Douglas. Detection of surface-specific points by local paprallel processing of discrete terrain elevation data. *Graphics Image Processing*, 4:475–387, 1975.
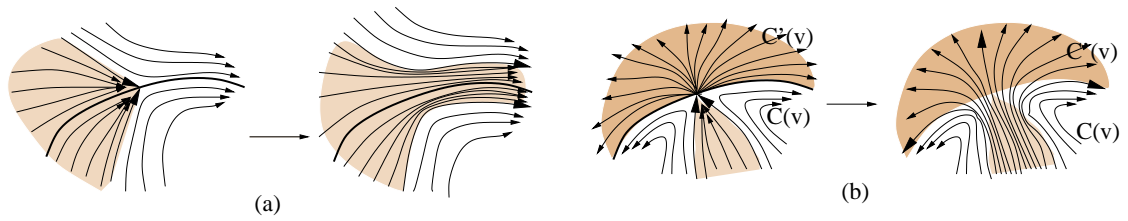
Figure 10: In (a), the canceling operation of an inflection point. In (b), component $C(v)$ is merged to component $C'(v)$.
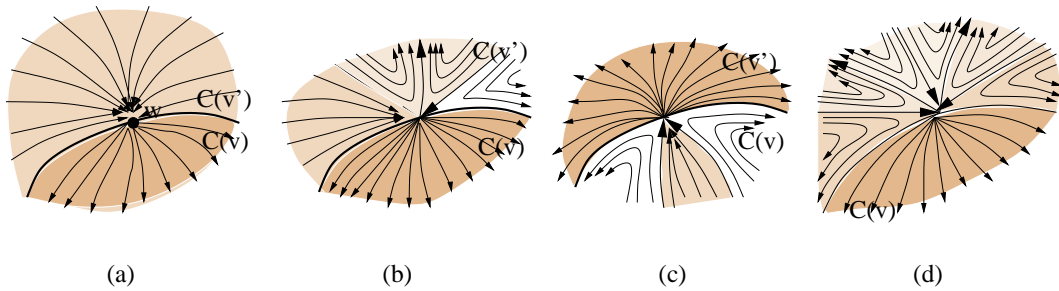

Figure 11: Cases (a) and (b) are impossible. Case in(c) illustrates an inflection point which can be canceled. Case in (d) illustrates a saddle point with one bundle which is $C(v)$.

[13] S. Smale. Morse inequalities for a dynamical system. *Bulletin of American Mathematical Society*, 66:43–49, 1960.

[14] R. Thom. Sur une partition en cellule associées a une fonction sur une variété. *C.R.A.S.*, 228:973–975, 1949.

[15] L. T. Watson, T. J. Laffey, and R.M. Haralick. Topographic classification of digital image intensity surfaces using generalised splines and the discrete cosine transformation. *Computer Vision, Graphics and Image Processing*, 29:143–167, 1985.
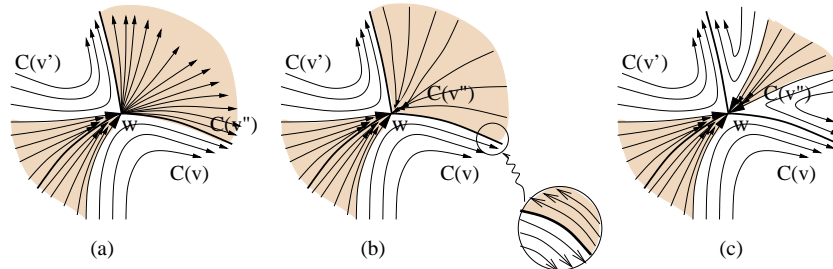
Figure 12: The nature of w common to three components. Point w is a slope point in both $C(v)$ and $C(v')$ and it becomes a saddle point in $C(v) \cup C(v')$. In (a) w is an inflection point in K. Case (b) is impossible: point w cannot be a minimum in $C(v")$. In (c), w is a saddle point in $C(v")$ and is a saddle in K.
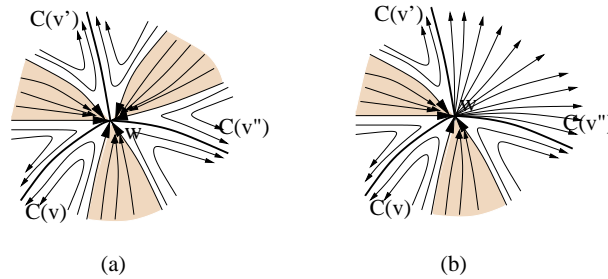

Figure 13: A Monkey saddle situation in (a): w is a saddle in each component. In (b) point w is saddle point in K.
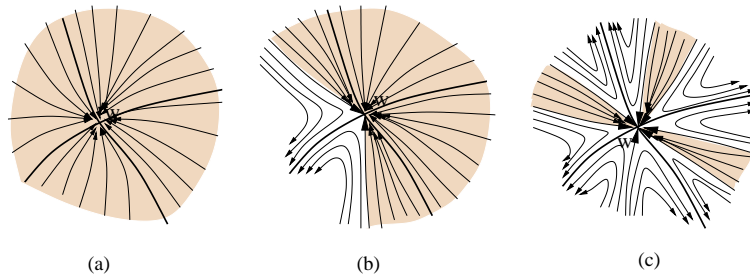

Figure 14: An example for the general situation. In (a) point w is a minimum in K. In (b) point w is an inflection point which can be eliminated. In (c) point w is a generalized saddle point with eight separatrices (three bundles and five l separatrices).
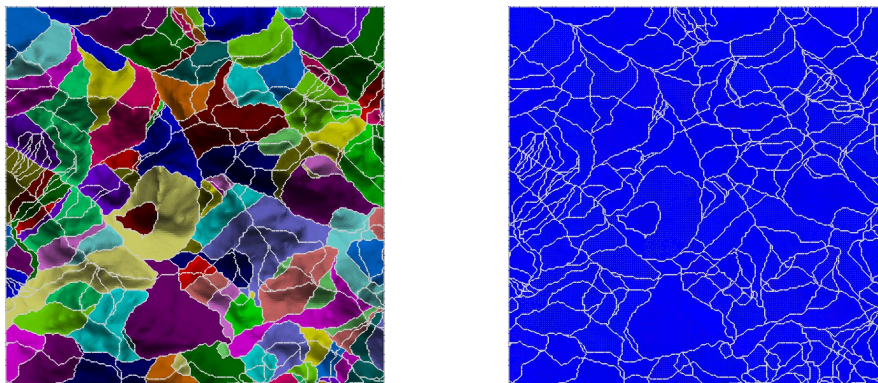

Figure 15: The critical net corresponding to the image of Mount Marcy already represented in Figure 3.