A Unified Interpolatory and Approximation $\sqrt{3}$ Subdivision Scheme

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Abstract

We have found that there is a relationship between the cubic B-spline and four-point curve subdivision method. In the paper it is used to deduce interpolatory subdivision schemes from cubic B-spline based approximation subdivision schemes directly and construct unified schemes for compositing approximation and interpolatory subdivision. A new interpolatory $\sqrt{3}$ subdivision scheme and a interpolatory and approximation blended $\sqrt{3}$ subdivision scheme are created by this straightforward method. The former produces C^1 limit surface and avoids the problem in the exsiting interpolatory $\sqrt{3}$ subdivision mask where the weight coefficients on extraordinary vertices can not be described by explicit formulation. The latter can be used to solve the "popping effect" problem when switching between meshes at different levels of resolution, provide the possibility to locally choose an interpolating variant of the conventionally approximating subdivision scheme, and give more flexibility for feature modeling. These are realized by only changing the value of a parameter. The method is thoroughly simple without needs of constructing and solving equations.

Categories and Subject Descriptors (according to ACM CCS): I.3.3 [Computer Graphics]: Surface modeling

1. Introduction

We have found that there are some relationships between the cubic B-spline and four-point curve subdivision method. This fact was first observed by Maillot and Stam [JM01]. However they were not aware of the property and haven't generalized it to extensive application. What's more, in their method, the way of computing positions of newly inserted edge vertices is not base on the theory of generalizing curve case to surface. It is the essential reason why the method can't produce C^1 limit interpolating surface.

To force the limit surface to go through a particular set of control points, modifications of the Catmull-Clark and Doo-Sabin schemes are needed. Nasri [Nas87] presents such a modification for the Doo-Sabin algorithm. Halstead et al. [MH93] propose an interpolation scheme using Catmull-Clark surfaces that minimizes a certain fairness measure. Both methods require the construction of a linear constraint on the control points for each interpolation point and thus the establishment of a system of linear equations. The initial mesh for the subdivision surface can be obtained by solving the equations. However, as pointed out in [MH93], it is possible for the coefficient matrix in the linear system to be singular, and it is unclear under what conditions the linear system is soluble.

"Popping effect" problem was also introduce in [JM01]. The limit surface of the approximating subdivision, especially in locally convex areas, is smaller than the base mesh. This is because the refined control meshes progressively shrink towards the limit surface. Consequently, noticeable "popping effects" occur when switching between meshes at different levels of resolution. On the other hand, Interpolating schemes suffer from the opposite problem. Limit surfaces tend to bulge out of the polygonal control mesh, and successive subdivision steps converge to a surface that is too big. We are aware that there are only two kinds of methods have been stated respect to the problem. In one kind of the work, it is achieved by changing the initial control mesh [MH93], not the subdivision rules themselves. It requires

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constructing and solving a linear equations which causes high computation complexity. In another work [JM01], it is realized by adding a push-back step where each original vertex is moved back towards its original position and newly introduced vertices are also adjusted by linear interpolation of the adjusted original vertices. As the reason we have analyzed before, it can't produces C^1 limit surface.

Base on the relationship between the cubic B-spline and four-point curve subdivision method, in the paper we will directly deduce a new interpolatory $\sqrt{3}$ subdivision mask for triangular mesh from $\sqrt{3}$ subdivision mask. The method is sufficiently straightforward without the additional needs of computing the mask on extraordinary vertices. The new interpolatory $\sqrt{3}$ subdivision mask avoids the problem in the exsiting interpolatory $\sqrt{3}$ subdivision mask [UL00] where the weight coefficients on extraordinary vertices can not be described by explicit formulation.

A unified scheme for compositing approximation and interpolatory $\sqrt{3}$ subdivision is also constructed. It is proposed by adding a parameter to control the subdivision surfaces to approach the control meshes. By changing the value of the parameter, it can solve the "popping effect" problem and force the limit surface to go through a particular set of control points. The method is thoroughly simple without any complicated computation.

2. $\sqrt{3}$ Subdivision

In this section we present $\sqrt{3}$ subdivision which was introduced in [Kob00] for an approximatory subdivision scheme. The splitting operator used for our subdivision is the same with it.

In $\sqrt{3}$ subdivision scheme in the middle of every triangle of a mesh a new vertex is computed. This vertex is connected to the old vertices of the triangle which perform a 1-to-3 split for the triangle. In order to re-balance the valence of the mesh vertices then flip every original edge that connects two old vertices as in figure 1.

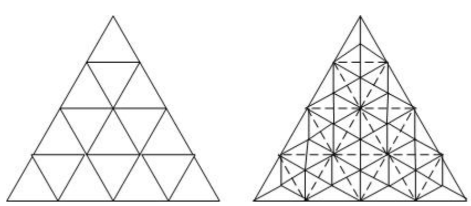

Figure 1: $\sqrt{3}$-*splitting of the mesh.*

In approximating $\sqrt{3}$ subdivision presented by Kobbelt the placement of the newly inserted vertices q is the center of the triangle $\triangle(P_i, P_j, P_k)$.

$$q := \frac{1}{3}(P_i + P_j + P_k) \tag{1}$$

The stencil for the relaxation of the old vertices is the 1-ring neighborhood containing the vertex itself and its direct neighbors. Let q be a old vertex of the mesh, then H(q) is the set containing all the vertices sharing an edge with q.

$$q := \left(1 - \frac{4-2\cos(\frac{2\pi}{n})}{9}\right)q + \frac{4-2\cos(\frac{2\pi}{n})}{9}\frac{1}{n}\sum_{Q\in H(q)} Q \tag{2}$$

3. A relationship between cubic B-spline curve and four-point curve

As is usually the case in subdivision researchs, first discover things in a curve setting and then generalize them to surfaces. Given the original control vertices $P_i^0 (i = 0, 1, 2...n)$, insert new vertex P_i^1 in the middle of vertices P_i^0 and P_{i+1}^0. In each step of cubic B-spline curve subdivision, the newly inserted points P_i^1 remain stable and the old points P_i^0 move to new palcements P_i^2. In each step of four point curve subdivision, the old points P_i^0 remain stable and the newly inserted points P_i^1 move to new palcements P_i^3. Let $\Delta(P_i) = P_i^2 - P_i^0$. It is concluded:

$$P_i^3 = P_i^1 - \frac{1}{2}(\Delta(P_i) + \Delta(P_{i+1})) \tag{3}$$

Equation (3) means that each step of four point curve subdivision can be gained from the corresponding step of cubic B-spline curve subdivision. The figure 2 shows this relation.

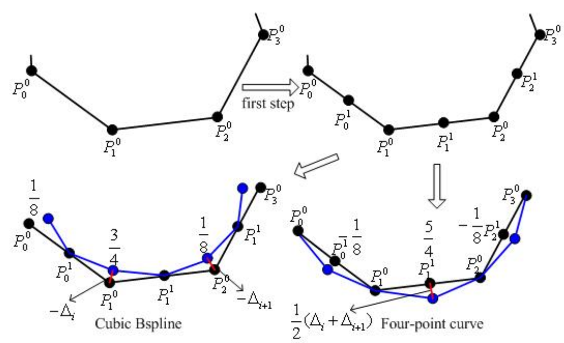

Figure 2: *Relevancy between the cubic B-spline and four-point curves subdivision.*

Now add a parameter α to control the subdivision surfaces to approach the control meshes. In order to force the limit surface to go through a particular set of control points, permit the parameter α of different vertex to be unequal. Thus, we get the following non-uniform subdivision scheme. Given the original control vertices P_i^0 and its weights α_i^0, the non-uniform unified scheme defines points at level $j+1$ of the recursion by:

$\alpha_{2i}^{j+1} = \alpha_i^j$

$\alpha_{2i+1}^{j+1} = \frac{1}{2}(\alpha_i^j + \alpha_{i+1}^j)$

$$P_{2i}^{j+1} := \frac{\alpha_{2i}^{j+1}}{8}P_{i-1}^{j} + \frac{4-\alpha_{2i}^{j+1}}{4}P_{i}^{j} + \frac{\alpha_{2i}^{j+1}}{8}P_{i+1}^{j}$$

$$P_{2i+1}^{j+1} := \frac{\alpha_{2i+1}^{j+1}-1}{16}P_{i-1}^{j} + \frac{9-\alpha_{2i+1}^{j+1}}{16}P_{i}^{j} + \frac{9-\alpha_{2i+1}^{j+1}}{16}P_{i+1}^{j} + \frac{\alpha_{2i+1}^{j+1}-1}{16}P_{i+2}^{j}$$

Figure 3: *Curve samples with different value of parameter.*

When $\alpha_i^0 \equiv \alpha$ where α is a constant, it is a uniform stationary subdivision scheme produce C^1 limit curve. When $\alpha_i^0 \in [0,1]$, it is a non-uniform subdivision scheme produce C^1 limit curve. The proof is shown in the appendix.

4. New Interpolatory $\sqrt{3}$ Subdivision

In this section we want to develop rules for interpolatory $\sqrt{3}$ subdivision scheme from the corresponding $\sqrt{3}$ subdivision scheme. Hence, the positions of old vertices remain stable. Only one rule is needed. Insert a new vertex in every triangle

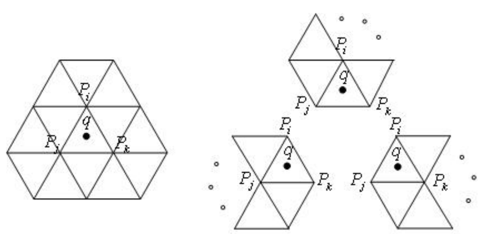

Figure 4: *Stencil for the computation of a new vertex. The left is the case of regular vertices and the right is the case of extraordinary vertices.*

$\triangle(P_i,P_j,P_k)$ of a mesh and the position of the new vertex q is computed by the formulae as follows:

$$q := \frac{1}{3}(P_i+P_j+P_k) + \frac{1}{3}(\Delta(P_i)+\Delta(P_j)+\Delta(P_k)) \quad (4)$$

$$= f(P_i)+f(P_j)+f(P_k) \quad (5)$$

$$f(P) = \frac{13-2\cos(\frac{2\pi}{n})}{27}P - \frac{4-2\cos(\frac{2\pi}{n})}{27}\frac{1}{n}\sum_{Q\in H(P)} Q \quad (6)$$

where $\Delta(P)$ is the displacement of the vertex P during one step of $\sqrt{3}$ Subdivision. Note that, the weight coefficients of this new interpolatory $\sqrt{3}$ Subdivision mask can been described in explicit formulation (5) wherever at regular vertices or extraordinary vertices.

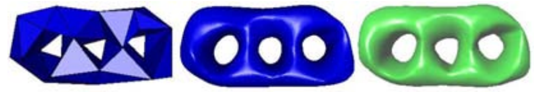

Figure 5: *Subdivision a simple hole model with valences 5,6,7,8 vertice(left), subdivided with modified Butterfly scheme(middle) and the new interpolatory $\sqrt{3}$-subdivision scheme(right).*

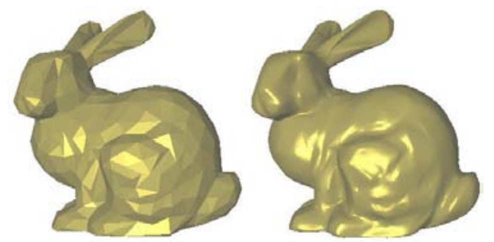

Figure 6: *Subdividing a bunny model with the new interpolatory $\sqrt{3}$-subdivision scheme.*

5. Approximation and interpolatory blended $\sqrt{3}$ Subdivision.

In this section, a unified scheme for compositing approximation and interpolatory $\sqrt{3}$ subdivision is proposed by adding a parameter α to control the subdivision surfaces to approach the control meshes. Two rules were used, one for inserting a new vertex in every triangle and a second for computing new positions of the already existing vertices. The position for newly inserted vertex q and its parameter weight α_q in triangle $\triangle(P_i,P_j,P_k)$ is computed by the formulae as follows:

$$\alpha_q := \frac{1}{3}(\alpha_i+\alpha_j+\alpha_k) \quad (7)$$

$$q := f(P_i,\alpha_i)+f(P_j,\alpha_j)+f(P_k,\alpha_k) \quad (8)$$

$$f(P,\alpha) = \frac{1}{3}(P+\alpha\Delta(P)) \quad (9)$$

$$= \frac{9+4\alpha-2\alpha\cos(\frac{2\pi}{n})}{27}P - \frac{4\alpha-2\alpha\cos(\frac{2\pi}{n})}{27}\frac{1}{n}\sum_{Q\in H(P)} Q \quad (10)$$

The formula for the relaxation of the old vertices q:

$$q := q-(1-\alpha_q)\Delta(q) \quad (11)$$

$$= \left(\frac{5+4\alpha_q+2(1-\alpha_q)\cos(\frac{2\pi}{n})}{9}\right)q + \frac{(1-\alpha_q)(4-2\cos(\frac{2\pi}{n}))}{9}\frac{1}{n}\sum_{i=0}^{n-1}P_i$$

The original value of the parameter α_i of each control point is given by the user. When $\alpha_i \equiv 0$, it produces $\sqrt{3}$ subdivision scheme. When $\alpha_i \equiv 1$, it produces the new $\sqrt{3}$ interpolatory subdivision scheme. When $\alpha_i \in (0,1)$, the shrinking or bulge between the limit mesh and original mesh are smaller than the approaching scheme or the interpolating scheme. This can be used to solve the "popping effect" problem when switching between meshes at different levels of resolution.

It also provide the possibility to locally choose an interpolating variant of the conventionally approximating subdivision scheme. Let α_i of the interpolated original control points be 1, α_i of the approached original control points be 0, examples are shown in figure 7.

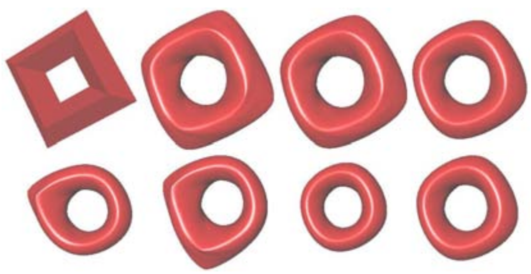

Figure 7: *Examples of our unified $\sqrt{3}$-subdivision scheme with different choices of α. The top row, from left to right, demonstrates control mesh, subdivided meshes with $\alpha \equiv 1.0$, $\alpha \equiv 0.8$, $\alpha \equiv 0.5$, respectively. The bottom row, from right to left, shows subdivided meshes with $\alpha \equiv 0.2$, $\alpha \equiv 0.0$, and two examples generated by interpolating some of the original control points only.*

6. Continuity Analysis

By definition, the subdivision matrix is a square matrix S which maps a certain sub-mesh $V \in M_k$ to a topologically equivalent submesh $S(V) \in M_{k+1}$ of the refined mesh. As have analyzed in [Kob00] [UL00], for the applied refinement operator it is not possible to directly use this matrix. This is because of a 30^0 rotation being performed when splitting the triangles. Therefore we subdivide the mesh twice leading to a 60^0 rotation which is corrected by resorting the vertices. This is done by multiplying the matrix with a permutation matrix R. So we heve to analyze the matrix: $\tilde{S} = RSS$.

When $\alpha_i^0 \equiv C$, the unified scheme is a uniform stationary subdivision scheme. Let $C \in [0,2]$ and take numerical verification by Matlab, the leading eigenvalue of each $\tilde{S}$ have following conditions:
For $n > 3$, $\lambda_0 = 1, \lambda_1 = \lambda_2 > \lambda_3 = \lambda_4$
For $n = 3$, $\lambda_0 = 1, \lambda_1 = \lambda_2 > \lambda_3$

According to [Rei95] [Zor97], it's a necessary condition for the convergence of the subdivision scheme to a C^1 continuous surface. The local regularity of the subdivision surface at extraordinary vertices requires the injectivity of the characterisitc map. One sample of isoparameter lines for these maps in the vicinity of irregular vertices are showed in figure 8. However, when the parameters α_i^0 have unequal values, the convergence analysis of such non-uniform schemes at extraordinary vertices is still an open question.

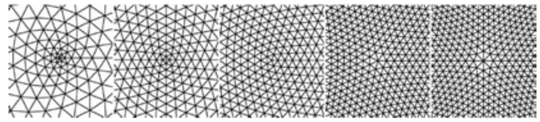

Figure 8: *Visualization of regularity and injectivity of characteristic maps for valence $n = 3,4,5,7,8$.*

7. Conclusions

In this paper, we have constructed a approximation and interpolatory blended subdivision scheme. It realizes the possibility to force the limit surface to go through a particular set of control points without needs of constructing and solving equations. It is also used to solve the "popping effect" problem and give more flexibility for feature modeling. Here we focus on $\sqrt{3}$ subdivision, however some other subdivision schemes can be treated in a similar way. The convergence analysis of the unified subdivision scheme in curve case is orbicularly proved, and the analysis for the mulitidimensional case of regular meshes can also be easily proved by this way [ND02]. However,the convergence analysis of such non-uniform schemes on extraordinary vertices is still an open question.

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