Appendix: Curve case

1. Unified interpolatory and approximation curve subdivision

Given the original control vertices $P_i^0 (i = 1,2...n)$, the unified scheme defines points at level $j+1$ of the recursion by:

$$\begin{cases} P_{2i}^{j+1} := \frac{\alpha}{8} P_{i-1}^j + \frac{4-\alpha}{4} P_i^j + \frac{\alpha}{8} P_{i+1}^j \\ P_{2i+1}^{j+1} := \frac{\alpha-1}{16} P_{i-1}^j + \frac{9-\alpha}{16} P_i^j + \frac{9-\alpha}{16} P_{i+1}^j + \frac{\alpha-1}{16} P_{i+2}^j \end{cases} \quad (1)$$

When $\alpha = 0$, the subdivision scheme produces cubic B-splines in the limit. On the other hand, when $\alpha = 1$ produces four-point curves in the limit.

In order to force the limit surface to go through a particular set of control points, permit the parameter α of each vertex to be unequal. Thus, we get the following non-uniform unified subdivision scheme.

Given the original control vertices P_i^0 and their weights $\alpha_i^0 (i = 1,2...n)$, the non-uniform unified scheme defines points at level $j+1$ of the recursion by:

$$\begin{cases} \alpha_{2i}^{j+1} := \alpha_i^j \\ \alpha_{2i+1}^{j+1} := \frac{1}{2}(\alpha_i^j + \alpha_{i+1}^j) \\ P_{2i}^{j+1} := \frac{\alpha_{2i}^{j+1}}{8} P_{i-1}^j + \frac{4-\alpha_{2i}^{j+1}}{4} P_i^j + \frac{\alpha_{2i}^{j+1}}{8} P_{i+1}^j \\ P_{2i+1}^{j+1} := \frac{\alpha_{2i+1}^{j+1}-1}{16} P_{i-1}^j + \frac{9-\alpha_{2i+1}^{j+1}}{16} P_i^j + \frac{9-\alpha_{2i+1}^{j+1}}{16} P_{i+1}^j \\ \qquad\qquad + \frac{\alpha_{2i+1}^{j+1}-1}{16} P_{i+2}^j \end{cases} \quad (2)$$

When $\alpha_i \equiv \alpha$, it is a stationary uniform subdivision scheme as same as the scheme defined in equation (1). When $\alpha_i^0 = 1$, the limit curve interpolates the control vertices P_i^0. Figure1 shows the influence of the parameter on the subdivided control vertices after several refinements.

2. Convergence and smoothness analysis

A well-known method for analyzing convergence and smoothness of curve subdivision schemes have been presented by Dyn and Levin [Dyn02]. Here, we used the method to proof that our uniform unified scheme is C^1, and proof that our non-uniform unified stationary scheme is C^1 when $\alpha_i \in [0,1]$.

2.1. Analysis of univariate uniform unified schemes

Following the framework of [Dyn02], a binary subdivision scheme is a convergent if there is a function $p: \mathbb{R} \to \mathbb{R}$ with the property that for any compact set $K \subset \mathbb{R}$, $\lim_{k\to\infty} \max_{i\in\mathbb{Z}\cap 2^k K} | p_i^k - p(2^{-k} i) | = 0$.

The refinement rule in equation (1) at refinement level k can be written of the form:

$p_i^{k+1} = \sum_{j\in\mathbb{Z}} a_{i-2j} p_i^k$ or simply as $p^{k+1} = S_a p^k$

The symbol of S_a is the Laurent polynomial

$$a(z) = \sum_i a_i z^i = \frac{\alpha-1}{16} z^{-3} + \frac{\alpha}{8} z^{-2} + \frac{9-\alpha}{16} z^{-1} + \frac{4-\alpha}{4} z^0 + \frac{9-\alpha}{16} z^1 + \frac{\alpha}{8} z^2 + \frac{\alpha-1}{16} z^3.$$

This can be written as

$$a(z) = \frac{1}{2}(1+z)^2 b(z),$$

where

$$b(z) = \frac{\alpha-1}{8} z^{-3} + \frac{1}{4} z^{-2} + \frac{3-\alpha}{4} z^{-1} + \frac{1}{4} z^0 + \frac{\alpha-1}{8} z^1.$$

By Corollary of [Dyn02], if S_b is contractive then S_a is C^1. Defining

$$\| S_b^{[l]} \|_\infty := \max \left\{ \sum_{j\in\mathbb{Z}} \left| b_{i-2^l j}^{[l]} \right| : 0 \le i < 2^l \right\},$$

where $b^{[l]}(z) := b(z) b(z^2) \cdots b(z^{2^{l-2}})$.

we find that

$$\| S_b^{[l]} \|_\infty := \max \left\{ \frac{\alpha-1}{8} + \frac{3-\alpha}{4} + \frac{\alpha-1}{8}, \frac{1}{4} + \frac{1}{4} \right\} = \frac{1}{2} < 1,$$

which shows that S_b is contractive. So, S_a is C^1.

2.2. Analysis of non-uniform unified schemes

A non-uniform linear binary subdivision scheme can be represented by a bi-infinite sequence of generating polynomials $\{ f_{(j,k)}(z) \}$ where each $f_{(j,k)}$ is the polynomial representing the scheme generating $p_j^k, k \ge 1, j \in \mathbb{Z}$. That is,

$$p_i^{k+1} = \sum_{i\in\mathbb{Z}} f_{(i,k),j-2i} p_i^k$$

where $f_{(j,k)}(z) = \sum_{m\in\mathbb{Z}} f_{(j,k),m} z^m$.

Following the framework of [Dyn02], define $\delta^{0}_{(j,k)}(z) \equiv f_{(j,k)}(z)$. Then all the non-uniform difference schemes $\left\{\delta^{r}_{(j,k)}(z)\right\}, r=1,\ldots,m$ defined recursively by

$$\delta^{r+1}_{(j,k)}(z) = 2^{r}\left(z\delta^{r}_{(j-1,k)}(z) - \delta^{r}_{(j,k)}(z)\right)\Big/(z^{2}-1)$$

are finite Laurent polynomials

The generating polynomials of the ℓ -iterated scheme, transforming values at level k directly to level $k+\ell$, are $\left\{f_{(j,k,\ell)}(z)\right\}$ defined recursively by

$$f_{(j,k,\ell)}(z) = \sum_{m} q_{(j,k+i,1),m} z^{m} q_{\left(\left[\frac{j-m}{2}\right],k,i\right)}(z^{2})$$

where $q_{(j,k,1)}(z) = q_{(j,k+1)}(z)$

and $f_{(j,k,i)}(z) = \sum_{m} f_{(j,k,i),m} z^{m}, i=1,\ldots,\ell.$

In the non-uniform scheme defined by equation (2), the corresponding generating polynomials can be written as

$$f_{(j,k)}(z) = \frac{1}{2}(1+z)^{2}\left(\frac{\alpha^{k}_{j}-1}{8}z^{-3} + \frac{1}{4}z^{-2} + \frac{3-\alpha^{k}_{j}}{4}z^{-1} + \frac{1}{4}z^{0} + \frac{\alpha^{k}_{j}-1}{8}z^{1}\right)$$

$$\delta^{2}_{(j,k)}(z) = \frac{\alpha^{k}_{j}-1}{8}z^{-3} + \frac{1+\alpha^{k}_{j}-\alpha^{k}_{j-1}}{4}z^{-2} + \frac{\alpha^{k}_{j-2}-1}{8}z^{1} + \frac{6+\alpha^{k}_{j}-4\alpha^{k}_{j-1}+\alpha^{k}_{j-2}}{8}z^{-1} + \frac{1-\alpha^{k}_{j-1}-\alpha^{k}_{j-2}}{4}z^{0}$$

$$\delta^{2}_{(j,k,2)}(z) = \sum_{m=-9}^{3} q_{m} z^{m} = \qquad (5)$$

$$\frac{(\alpha^{k+2}_{j}-1)(\alpha^{k+1}_{i+1}-1)}{64}z^{-9} + \frac{(1+\alpha^{k+2}_{j}-\alpha^{k+2}_{j-1})(\alpha^{k+1}_{i+1}-1)}{32}z^{-8}$$

$$+\left[\frac{(6+\alpha^{k+2}_{j}-4\alpha^{k+2}_{j-1}+\alpha^{k+2}_{j-2})(\alpha^{k+1}_{i}-1)}{64} + \frac{(\alpha^{k+2}_{j}-1)(1+\alpha^{k+1}_{i+1}-\alpha^{k+1}_{i})}{32}\right]z^{-7} +$$

$$\left[\frac{(1+\alpha^{k+2}_{j}-\alpha^{k+2}_{j-1})(1+\alpha^{k+1}_{i+1}-\alpha^{k+1}_{i})}{16} + \frac{(1-\alpha^{k+2}_{j-1}-\alpha^{k+2}_{j-2})(\alpha^{k+1}_{i}-1)}{32}\right]z^{-6} +$$

$$\left[\frac{(\alpha^{k+2}_{j}-1)(6+\alpha^{k+1}_{i+1}-4\alpha^{k+1}_{i}+\alpha^{k+1}_{i-1})}{64} + \frac{(6+\alpha^{k+2}_{j}-4\alpha^{k+2}_{j-1}+\alpha^{k+2}_{j-2})(1+\alpha^{k+1}_{i}-\alpha^{k+1}_{i-1})}{32} + \frac{(\alpha^{k+2}_{j-2}-1)(\alpha^{k+1}_{i-1}-1)}{64}\right]z^{-5}$$

$$\left[\frac{(1+\alpha^{k+2}_{j}-\alpha^{k+2}_{j-1})(6+\alpha^{k+1}_{i+1}-4\alpha^{k+1}_{i}+\alpha^{k+1}_{i-1})}{32} + \frac{(1-\alpha^{k+2}_{j-1}-\alpha^{k+2}_{j-2})(1+\alpha^{k+1}_{i}-\alpha^{k+1}_{i-1})}{16}\right]z^{-4}$$

$$\left[\frac{(\alpha^{k+2}_{j}-1)(1-\alpha^{k+1}_{i}-\alpha^{k+1}_{i-1})}{32} + \frac{(6+\alpha^{k+2}_{j}-4\alpha^{k+2}_{j-1}+\alpha^{k+2}_{j-2})(6+\alpha^{k+1}_{i}-4\alpha^{k+1}_{i-1}+\alpha^{k+1}_{i-2})}{64} + \frac{(\alpha^{k+2}_{j-2}-1)(1+\alpha^{k+1}_{i-1}-\alpha^{k+1}_{i-2})}{32}\right]z^{-3} +$$

$$\left[\frac{(1+\alpha^{k+2}_{j}-\alpha^{k+2}_{j-1})(1-\alpha^{k+1}_{i}-\alpha^{k+1}_{i-1})}{16} + \frac{(1-\alpha^{k+2}_{j-1}-\alpha^{k+2}_{j-2})(6+\alpha^{k+1}_{i}-4\alpha^{k+1}_{i-1}+\alpha^{k+1}_{i-2})}{32}\right]z^{-2} +$$

$$\left[\frac{(6+\alpha^{k+2}_{j}-4\alpha^{k+2}_{j-1}+\alpha^{k+2}_{j-2})(1-\alpha^{k+1}_{i-1}-\alpha^{k+1}_{i-2})}{32} + \frac{(\alpha^{k+2}_{j}-1)(\alpha^{k+1}_{i-1}-1)}{64} + \frac{(\alpha^{k+2}_{j-2}-1)(6+\alpha^{k+1}_{i-1}-4\alpha^{k+1}_{i-2}+\alpha^{k+1}_{i-3})}{64}\right]z^{-1} +$$

$$\left[\frac{(1+\alpha^{k+2}_{j}-\alpha^{k+2}_{j-1})(\alpha^{k+1}_{i-1}-1)}{32} + \frac{(1-\alpha^{k+2}_{j-1}-\alpha^{k+2}_{j-2})(1-\alpha^{k+1}_{i-1}-\alpha^{k+1}_{i-2})}{16}\right]z^{0} +$$

$$\left[\frac{(\alpha^{k+2}_{j-2}-1)(1-\alpha^{k+1}_{i-2}-\alpha^{k+1}_{i-3})}{32} +\right.$$

$$+\frac{(6+\alpha_j^{k+2}-4\alpha_{j-1}^{k+2}+\alpha_{j-2}^{k+2})(\alpha_{i-2}^{k+1}-1)}{64}\Bigg]z^1$$

$$+\frac{(1-\alpha_{j-1}^{k+2}-\alpha_{j-2}^{k+2})(\alpha_{i-2}^{k+1}-1)}{64}z^2$$

$$+\frac{(\alpha_{j-2}^{k+2}-1)(\alpha_{i-3}^{k+1}-1)}{64}z^3$$

where $i=[j/2]$. By Corollary of [Dyn02], to check if the scheme $\{f_{(j,k)}(z)\}$ is C^1 it is enough to prove that $\{\delta_{(j,k)}^r(z)\}$ is contractive, and it is enough to show that there exists $\rho\in[0,1)$ such that each of the following four inequalities holds for any j and k: $|q_{-9}|+|q_{-5}|+|q_{-1}|+|q_3|\le\rho$, $|q_{-8}|+|q_{-4}|+|q_0|\le\rho$, $|q_{-7}|+|q_{-3}|+|q_1|\le\rho$ and $|q_{-8}|+|q_{-4}|+|q_0|\le\rho$.

To simplify the notation let us denote the parameters in (5) as $t_m=\alpha_{i-4+m}^{k+1}, m=1,\ldots,5$ and $t_m=\alpha_{j-8+m}^{k+2}, m=6,7,8$. Now we have to show that above four inequalities are satisfied for some fixed $\rho\in[0,1)$ for any $\alpha_j^k\in[0,1]$. The four inequalities take the form:

$$\frac{1}{64}\big[(t_8-1)(4+2t_5-4t_4+2t_3)+(t_6-1)(4+2t_3-4t_2+2t_1)$$
$$+2(6+t_8-4t_7+t_6)(1+t_4-2t_3-t_2)\big]\le\rho \qquad (6)$$

$$\frac{1}{32}\big[(1+t_8-t_7)(4+2t_5-4t_4+2t_3)$$
$$+2(1-t_7-t_6)(2+t_4-2t_3-t_2)\big]\le\rho \qquad (7)$$

$$\frac{1}{64}\big[t_8(8+2t_5-2t_4-6t_3+2t_2)+t_6(8+2t_4-2t_3-2t_2-2t_1)$$
$$+2(1-t_7)(4+2t_4-4t_3+2t_2)-2t_5+8t_4-8t_3+8t_2+2t_1\big]\le\rho \qquad (8)$$

$$\frac{1}{32}\big[2(1+t_8-t_7)(2+t_5-2t_4-t_3)$$
$$+(1-t_7-t_6)(4+2t_4-4t_3+2t_2)\big]\le\rho \qquad (9)$$

The inequalities (6)(7)(8) and (9) are easy satisfied with $\rho=\frac{7}{8}$ for any $t_m\in[0,1]$.

Thus, the scheme $\{f_{(j,k)}(z)\}$ is C^1.

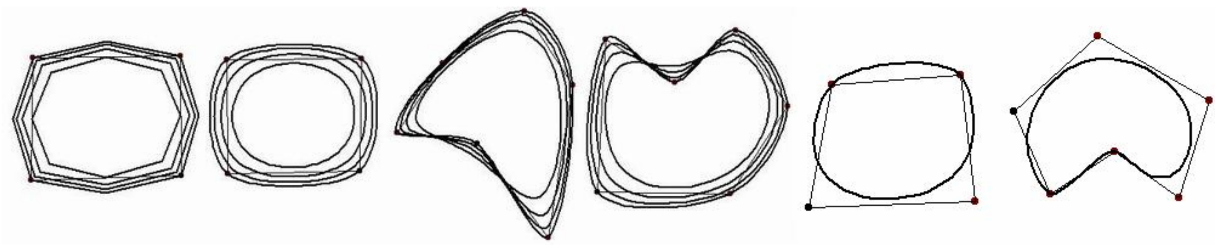

Figure1 Curve samples of our unified schemes with different value of parameter. From the inside curve to the outside curve where $\alpha=1;\alpha=0.5;\alpha=0.2;\alpha=0$ respectively. The first figure is subdivided once, the others are subdivided quartic. The right two are samples forcing the limit surface to go through a particular set of control points whose $\alpha_i^0=1$.