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Projective Geometry, Duality and Precision of Computation in Computer Graphics, Visualization and Games

Tutorial

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Plzen (Pilsen) City

Plzen is an old city [first records of Plzen castle 976] city of culture, industry, and brewery.

City, where today's beer fermentation process was invented
that is why today's beers are called Pilsner - world wide

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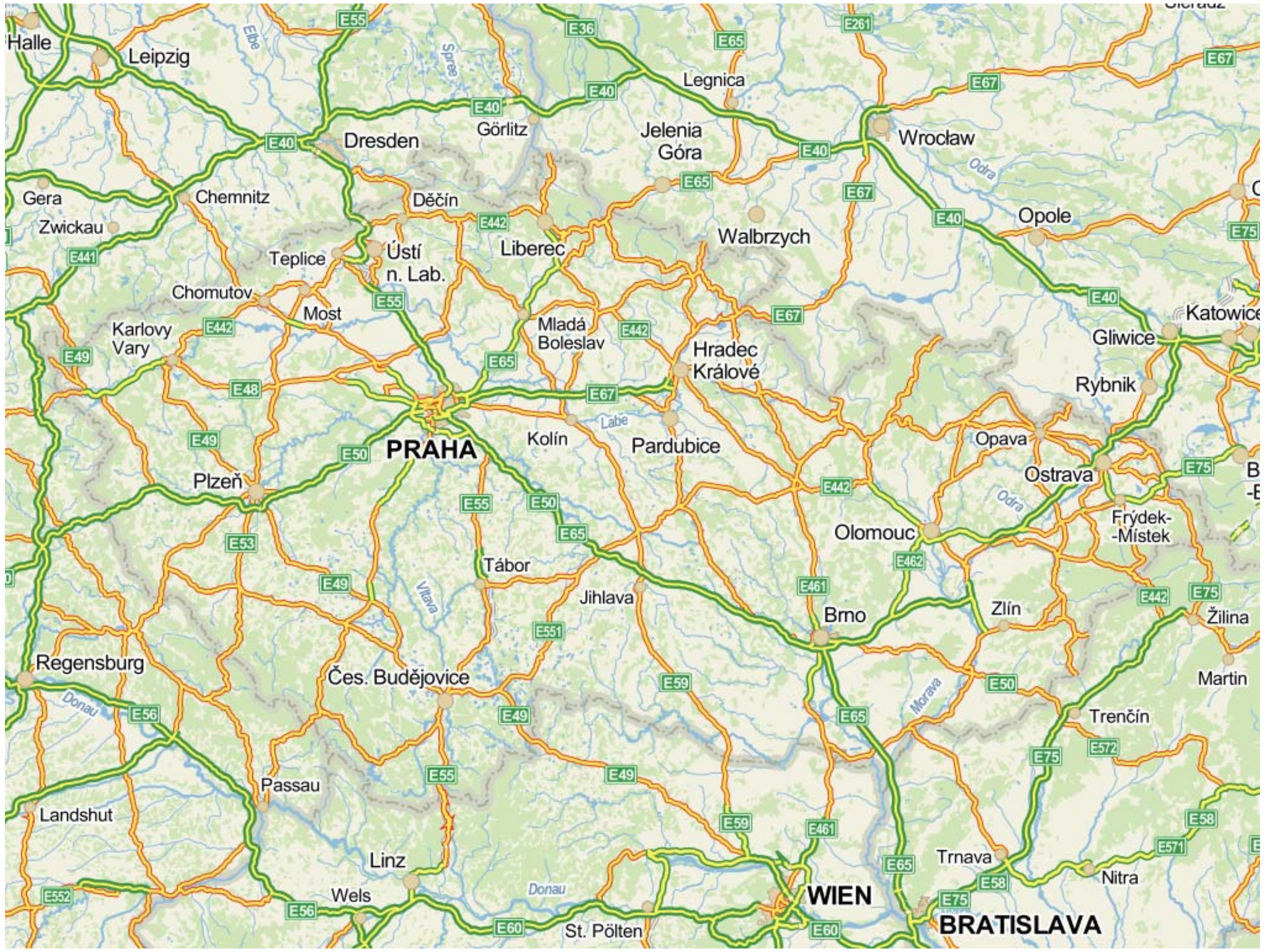
Ostrava City

Ostrava is

- an industrial city of coal mining & iron melting
- 3rd largest city



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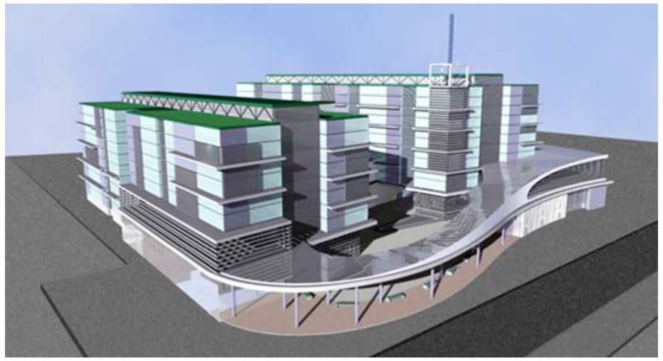
University of West Bohemia 17530 students + 987 PhD students

Computer Science and Engineering **Mathematics** (+ Geomatics)

Physics **Cybernetics** **Mechanics** (Computational)

- Over **50%** of income from research and application projects
- New research center EU project - investment 64 mil. EUR
- 2nd in the ranking of Czech technical / informatics faculties 2009, 2012



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An overview

- Precision and robustness
- Euclidean space and projective extension
- Principle of duality and its applications
- Geometric computation in the projective space
- Intersection of two planes in E3 with additional constrains
- Barycentric coordinates and intersections
- Interpolation and intersection algorithms
- Implementation aspects and GPU
- Conclusion and summary

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Numerical systems

- Binary system is used nearly exclusively
- Octal & hexadecimal representation is used
- If we would be direct descendants of tetrapods – we would have a great advantage – “simple counting in hexadecimal system”

	Name	Base	Digits	E min	E max
BINARY					
B 16	Half	2	10+1	−14	15
B 32	Single	2	23+1	−126	127
B 64	Double	2	52+1	−1022	1023
B 128	Quad	2	112+1	−16382	16383
DECIMAL					
D 32		10	7	−95	96
D 64		10	16	−383	384
D 128		10	34	−6143	6144

IEEE 758-2008 standard

ırtesy Clive "Max" Maxfield and Alvin Brown
tetrapods had eight fingers on each hand

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Mathematically perfect algorithms fail due to instability

Main issues

- stability, robustness of algorithms
- acceptable speed
- linear speedup – results depends on HW, CPU …. parameters !

Numerical stability

- limited precision of float / double
- tests A ? B with floats

  if A = B **then** ….. **else** …..   ;   **if** A = 0 **then** ….. **else** ….
  should be forbidden in programming languages

- division operation should be removed or postponed to the last moment if possible - “blue screens”, system resets

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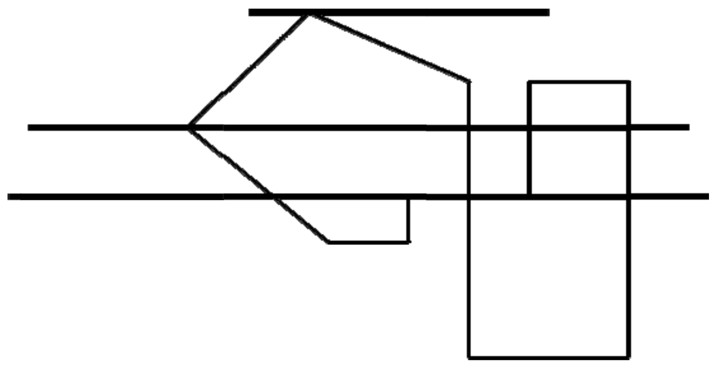
Typical examples of instability

- intersection of 2 lines in E3
- point lies on a line in E2 or a plane in E3

  $Ax + By + C = 0$ or $Ax + By + Cz + D = 0$
- detection if a line intersects a polygon, touches a vertex or passes through



Typical problem

```
double x = -1; double p = ....;
while ( x < +1)
{   if (x == p) Console.Out.WriteLine(” *** ”)
    x += p;
}
/*    if p = 0.1 then no output,   if p = 0.25 then expected output */
```

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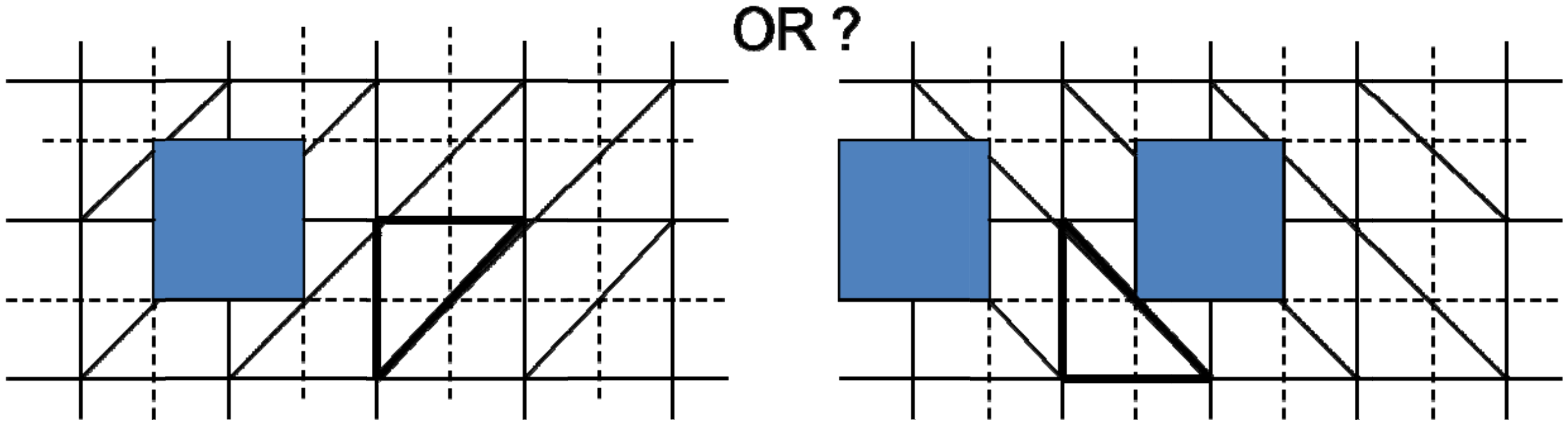
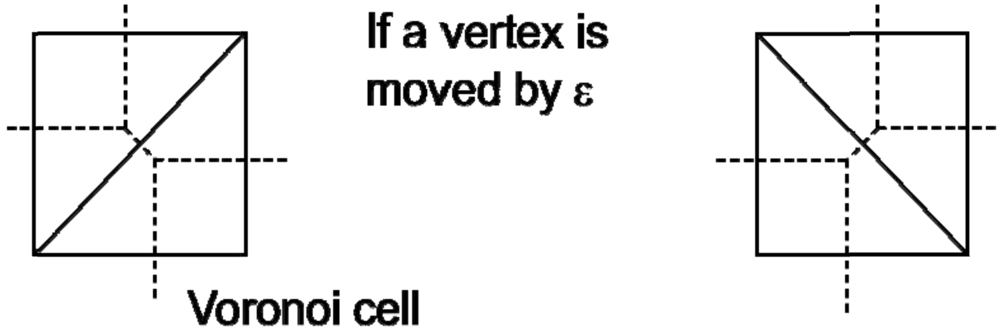
Delaunay triangulation & Voronoi diagram

Point inside of a circle given by three points – problems with meshing points in regular rectangular grid.



It can be seen that the DT & VD is very sensitive to a point position change

?? ROBUSTNESS ??



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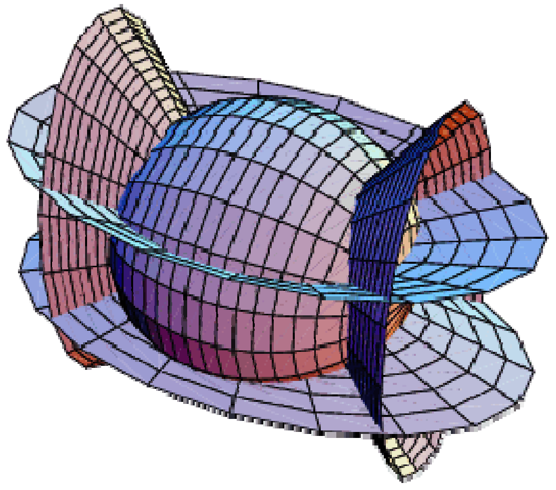
Vectors and Points in Geometry


Courtesy of
http://mathworld.wolfram.com/
ConfocalEllipsoidalCoordinates.html

Vectors – movable, no fixed position

- Points – no size, position fixed in the GIVEN coordinate system

Coordinate systems:

- Cartesian – left / right handed
  right handed system is used
- Polar
- Spherical, Cylindrical
- and many others, e.g. Confocal Ellipsoidal Coordinates
  http://mathworld.wolfram.com/ConfocalEllipsoidalCoordinates.html

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Floating point

- Not all numbers are represented correctly
- Logarithmic arithmetic
- Continuous fractions
- Interval arithmetic

$$\pi = \cfrac{4}{1+\cfrac{1^2}{3+\cfrac{2^2}{5+\cfrac{3^2}{\dots}}}}$$

$$\pi = [3;7,15,1,292,1,1,1,2,1,3,1 \dots]$$

Numerically NOT valid identities due to limited precision

- $cos^2\alpha + cos^2\beta = 1$
- $x^2 - y^2 = (x-y)(x+y)$

x + y = [a + c, b + d]   x = [a , b]

x - y = [a - d, b - c]   y = [c , d]

x × y = [min(ac, ad, bc, bd), max(ac, ad, bc, bd)]

x / y = [min(a/c, a/d, b/c, b/d),

max(a/c, a/d, b/c, b/d)] if y ≠ 0

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Statements like

if <float> = <float> **then ….** or **if** <float> ≠ <float> **then ….**

should not be allowed in programming languages

Quadratic equation - more reliable results

$at^2 + bt + c = 0$ usually solved as $t_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

If $b^2 \gg 4ac$ then

$q = -(b + sign(b)\sqrt{b^2 - 4ac}\,)/2$ $t_1 = {}^{q}/_{a}$ $t_2 = {}^{c}/_{a}$

The discriminant should be computed with a twice precision

Vieta's formula $t_1 + t_2 = -\,{}^{b}/_{a}$ $t_1 t_2 = {}^{c}/_{a}$

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Function value computation at $x = 77617, \; y = 33096$

$$f(x,y) = 333.75y^6 + x^2(11x^2y^2 - y^6 - 121y^4 - 2) + 5.5y^8 + x/(2y)$$

$f = 6.33835\ 10^{29}$ single precision

$f = 1{,}1726039400532$ double precision

$f = 1{,}172603940053178631858834904520183{8}$ extended precision

The correct result is “somewhere” in the interval of

$[-0{,}827396059946821368141165095479816292\mathbf{005},$
$-0{,}827396059946821368141165095479816291\mathbf{986}]$

Exact solution

$$f(x,y) = -2 + \frac{x}{2y} = \frac{54767}{66192}$$

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Summation is one of often used computations.

$$\sum_{i=1}^{10^3} 10^{-3} = 0.999990701675415$$

or

$$\sum_{i=1}^{10^4} 10^{-4} = 1.000053524971008$$

The result should be only one.

The correctness in summation is very important in power series computations.
!!!!ORDER of summation

$$\sum_{n=1}^{10^6} \frac{1}{n} = 14.357357 \qquad \sum_{n=10^6}^{1} \frac{1}{n} = 14.392651$$

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Recursion

Towers of Hanoi

```
MOVE (A, C, n);
{   MOVE (A, B, n-1);
    MOVE (A, C, 1);
    MOVE (B, C, n-1)
}  # MOVE (from, to, number) #
```

Ackermann function

$$A(m,n) = \begin{cases} n+1 & if\ m=0 \\ A(m-1,1) & if\ M>0\ and\ n=0 \\ A\big(m-1, A(m,n-1)\big) & if\ m>0\ and\ N>0 \end{cases}$$

The value of the function grows very fast as

$$A(4,4) = 2^{2^{2^{65536}}} = 2^{2^{10^{197296}}}$$

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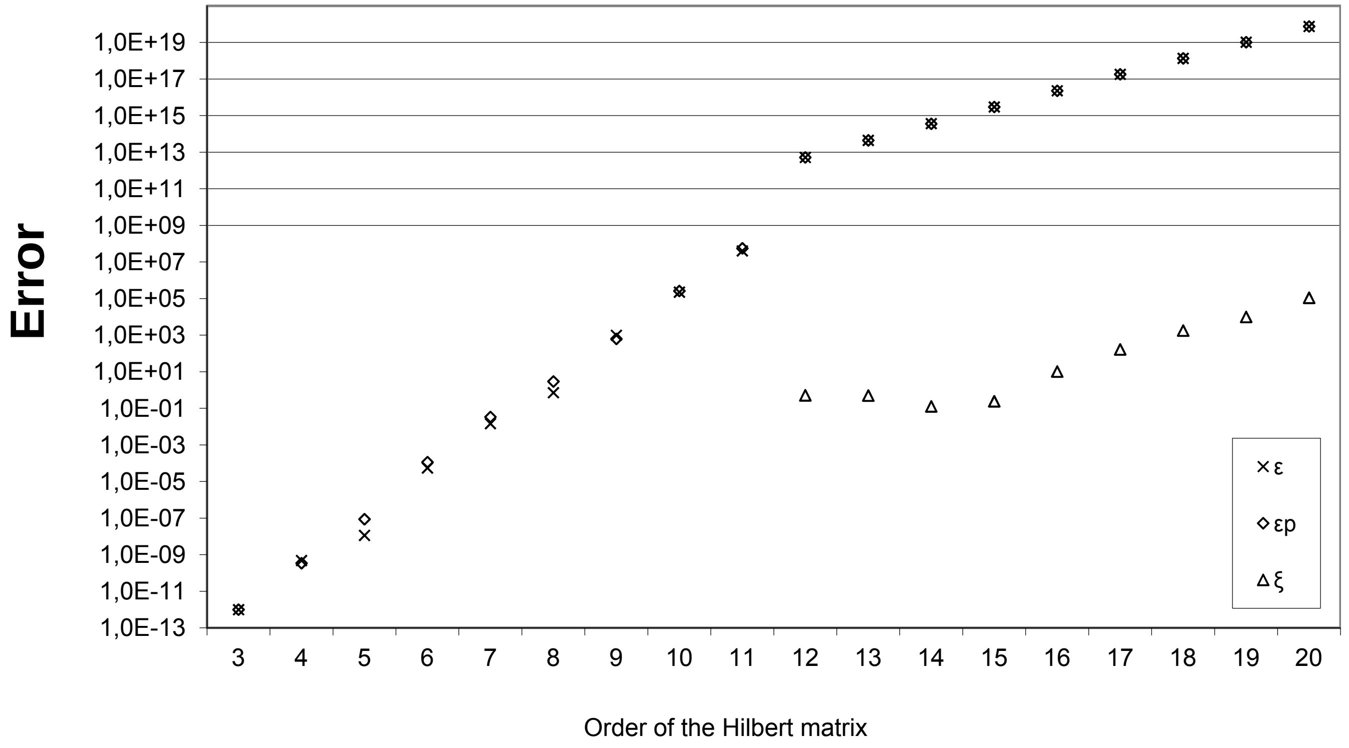
Numerical computations $\boldsymbol{Ax = b}$ $\boldsymbol{x = A^{-1}b}$

Hilbert's Matrix Inversion $H_{ij} = \frac{1}{i+j-1}$

$$H_{ij}^{-1} = (-1)^{i+j}(i+j-1)\binom{n+i-1}{n-j}\binom{n+j-1}{n-i}\binom{i+j-2}{i-1}^2$$



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Mathematical "forms" **There are several "forms":**

Implicit $F(x,y,z)=0$ or $F(\boldsymbol{x})=0$ or $\boldsymbol{F}(\boldsymbol{x})=\mathbf{0}$ (system of equations)

There is no orientation, e.g.

- if $F(\boldsymbol{x})=0$ is a iso-curve there is no hint how to find another point of this curve, resp. a line segment approximating the curve => tracing algorithms
- if $F(\boldsymbol{x})=0$ is a iso-surface there is no hint how to find another point of this surface => iso-surface extraction algorithms

Parametrical $\boldsymbol{x}=\boldsymbol{x}(u)$ $\boldsymbol{x}=\boldsymbol{x}(u,v)$

Points of a curve are "ORDERED" according to a parameter u, resp. u,v

Explicit $z=f(x)$ $z=f(x,y)$ [actually 2 ½ D]

For the given value x, resp. x,y we get function value z

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Implicit form

- Is used for separation - for detection if a point is inside or outside, e.g. a half-plane or a circle etc.
- There is always a question how to compute $\boldsymbol{x}$ of $F(\boldsymbol{x}) = 0$ as there are several solutions in general
- Complexity of computations × precision of computation

Compiler optimization is **DANGEROUS** in general - numerical precision

$$x^2 - y^2 = (x+y)(x-y)$$

$$\begin{vmatrix} A_x & A_y & A_x^2 + A_y^2 & 1 \\ B_x & B_y & B_x^2 + B_y^2 & 1 \\ B_x & B_y & B_x^2 + B_y^2 & 1 \\ B_x & B_y & B_x^2 + B_y^2 & 1 \end{vmatrix} = \begin{vmatrix} A_x - D_x & A_y - D_y & (A_x^2 - D_x^2) + (A_y^2 - D_x^2) \\ B_x - D_x & B_y - D_y & (B_x^2 - D_x^2) + (B_y^2 - D_x^2) \\ C_x - D_x & C_y - D_y & (C_x^2 - D_x^2) + (C_y^2 - D_x^2) \end{vmatrix} > 0$$

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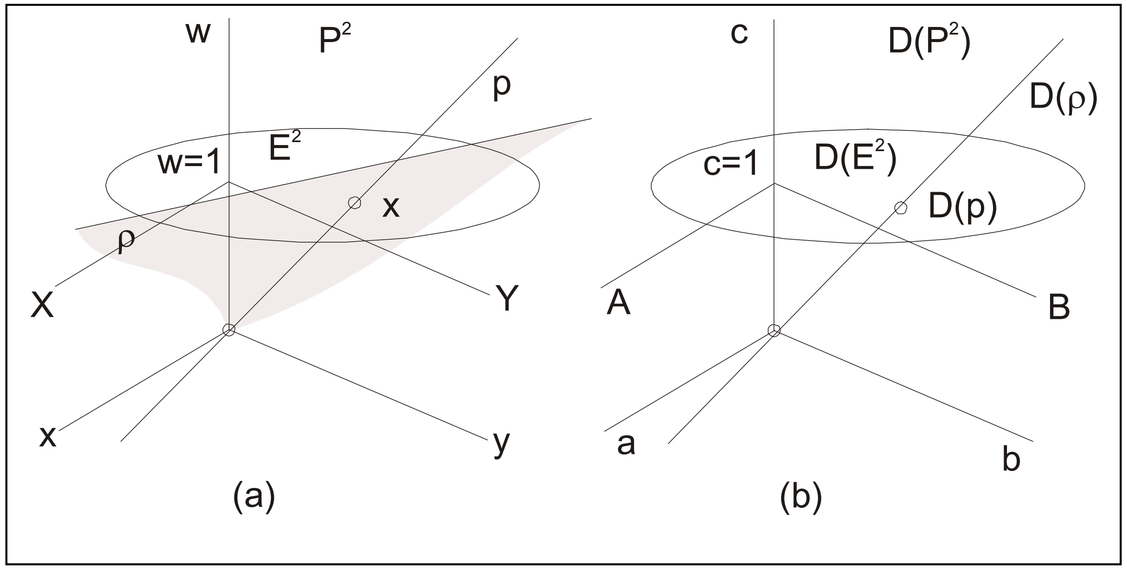
Projective Space

$\boldsymbol{X} = [X, Y]^T \qquad \boldsymbol{X} \in E^2$

$\boldsymbol{x} = [x, y{:}\, w]^T \qquad \boldsymbol{x} \in P^2$

Conversion:

$\boldsymbol{X} = [x/w, y/w]^T \qquad w \neq 0$



If $w = 0$ then $\boldsymbol{x}$ represents "an ideal point" - a point in infinity, i.e. it is a directional vector.

The Euclidean space E^2 is represented as a plane $w = 1$.

Equivalent "mathematical" notation often used:

$\boldsymbol{x} = [w{:}\, x, y]^T$ generally for E^n $\boldsymbol{x} = [x_0{:}\, x_1, \dots, x_n]^T$

i.e. homogeneous coordinate is the first

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Points and vectors

- Vectors **are "freely movable"** – not having a fixed position

$$\boldsymbol{a}_1 = [x_1, y_1 : 0]^T$$

- Points are **not "freely movable"** – they are fixed to an origin of the current coordinate system

$$\boldsymbol{x}_1 = [x_1, y_1 : w_1]^T \quad \text{and} \quad \boldsymbol{x}_2 = [x_2, y_2 : w_2]^T$$

usually in textbooks $w_1 = w_2 = 1$

A vector $\boldsymbol{A} = \boldsymbol{X}_2 - \boldsymbol{X}_1$ in the Euclidean coordinate system - **CORRECT**

Horribleconstruction" DO NOT USE IT – IT IS TOTALLY WRONG

$$\boldsymbol{a} = \boldsymbol{x}_2 - \boldsymbol{x}_1 = [x_2 - x_1, y_2 - y_1 : w_2 - w_1]^T = [x_2 - x_1, y_2 - y_1 : 1 - 1]^T$$
$$= [x_2 - x_1, y_2 - y_1 : 0]^T$$

This was presented as "How a vector" is constructed in the projective space P^k in a textbook!! **WRONG, WRONG, WRONG**

This construction has been found in SW as well!!

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Points and vectors

A Euclidean vector given by two points in the homogeneous coordinates

$$\boldsymbol{a} = \boldsymbol{x}_2 - \boldsymbol{x}_1 = [w_1 x_2 - w_2 x_1, w_1 y_2 - w_2 y_1 : w_1\, w_2]^T$$

This is the **CORRECT SOLUTION**, but what is the interpretation?

A "difference" of coordinates of two points is a vector in the mathematical meaning and $w_1\, w_2$ is a "scaling" factor actually

Actually the division operation is postponed and not performed immediately. A vector in projective notation

$$\boldsymbol{a} = \boldsymbol{x}_2 - \boldsymbol{x}_1 = [w_1 x_2 - w_2 x_1, w_1 y_2 - w_2 y_1 : w_1\, w_2]^T$$
$$\triangleq \left[\frac{w_1 x_2 - w_2 x_1}{w_1\, w_2}, \frac{w_1 y_2 - w_2 y_1}{w_1\, w_2} : 0\right]^T$$

where: $\triangleq$ means projectively equivalent

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A Euclidean vector in the projective representation (if the vector length matters)

$$\boldsymbol{a} = \boldsymbol{x}_2 - \boldsymbol{x}_1 = [w_1 x_2 - w_2 x_1, w_1 y_2 - w_2 y_1 : w_1\, w_2]^T$$
$$\triangleq \left[\frac{w_1 x_2 - w_2 x_1}{w_1\, w_2}, \frac{w_1 y_2 - w_2 y_1}{w_1\, w_2} : 0\right]^T$$

A vector in the projective space is given by coordinates x, y, w as

$$\boldsymbol{a} = \boldsymbol{x}_2 - \boldsymbol{x}_1 = [x_2 - x_1, y_2 - y_1, w_2 - \ w_1]^T$$

[=>Linear interpolation with a non-linear monotonic interpolation]

We have to strictly distinguish meaning of one *dimensional array* [*vector*], i.e. if we are working with:

- **points**, i.e. a data structure represent point coordinates, or
- **vectors**, i.e. a data structure represent a vector in the mathematical meaning

VECTORS x POINTS

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Duality

For simplicity, let us consider a line p defined as:

$$aX + bY + c = 0$$

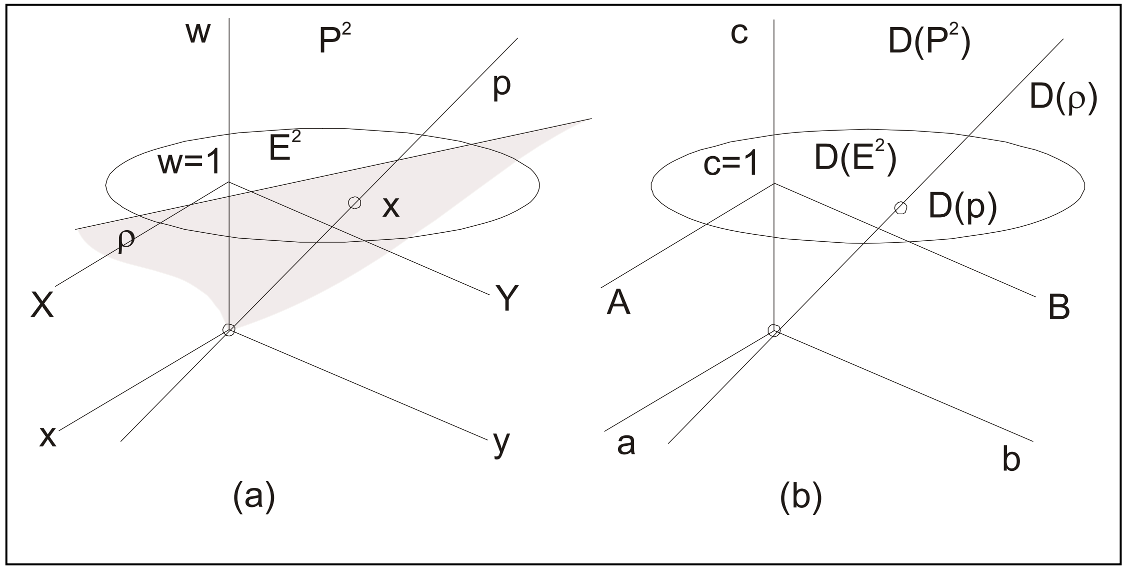
We can multiply it by $w \neq 0$ and we get:

$$awX + bwY + cw = 0$$

As $x = wX$ and $y = wY$

$ax + by + cw = 0$ i.e. $\boldsymbol{p}^T\boldsymbol{x} = 0$

$\boldsymbol{p} = [a, b: c]^T$ $\boldsymbol{x} = [x, y: w]^T = [wX, wY: w]^T$



A line $p \in E^2$ is actually a plane in the projective space P^2 (point $\boldsymbol{x} = [0,0: 0]^T$ excluded)

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Duality

From the mathematical notation $\boldsymbol{p}^{\boldsymbol{T}}\boldsymbol{x} = 0$

we cannot distinguish whether $\boldsymbol{p}$ is a line and $\boldsymbol{x}$ is a point or vice versa in the case of P^2. It means that

- a *point* and a *line* **are dual** in the case of P^2, and
- a *point* and a *plane* **are dual** in the case of P^3.

The principle of duality in P^2 states that:

Any theorem in E^2 remains true when we interchange the words "point" and "line", "lie on" and "pass through", "join" and "intersection", "collinear" and "concurrent" and so on.

Similarly for the E^3 case.

Once the theorem has been established, the dual theorem is obtained as described above.

This helps a lot to solve some geometrical problems.

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Examples of dual objects and operators

	Primitive	**Dual primitive**
P^2	Point	Line
	Line	Point
P^3	Point	Plane
	Plane	Point

Operator	**Dual operator**
Join	Intersect
Intersect	Join

Computational sequence for a problem is the same for a dual problem.

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Intersection of two lines

Let two lines p_1 and p_2 are given by

$$p_1 = [a_1, b_1 : c_1]^T \qquad \text{and} \qquad p_2 = [a_2, b_2 : c_2]^T$$

We have to solve a system of linear equations $\boldsymbol{Ax} = \boldsymbol{b}$

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} -c_1 \\ -c_2 \end{bmatrix} \; *$$

Then well known formula is used

$$x = \frac{Det_x}{Det} = \frac{det \begin{bmatrix} q_1 & b_1 \\ q_2 & b_2 \end{bmatrix}}{det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}} \qquad y = \frac{Det_x}{Det} = \frac{det \begin{bmatrix} a_1 & q_1 \\ a_2 & q_2 \end{bmatrix}}{det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}}$$

But what if Det is small?

Usually a sequence like $\boldsymbol{if}\ abs(\det(..)) \leq eps\ \boldsymbol{then}$ is used. What is eps?

Note * usually a line is in its explicit form as $ax + by = q$ instead of $ax + by + c = 0$, i.e. the implicit form

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Definition

The cross product of the two vectors

$$\boldsymbol{x}_1 = [x_1, y_1 : w_1]^T \text{ and } \boldsymbol{x}_2 = [x_2, y_2 : w_2]^T$$

is defined as:

$$\boldsymbol{x}_1 \times \boldsymbol{x}_2 = det \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x_1 & y_1 & w_1 \\ x_2 & y_2 & w_2 \end{bmatrix}$$

where: $\boldsymbol{i} = [1,0:0]^T \quad \boldsymbol{j} = [0,1:0]^T \quad \boldsymbol{k} = [0,0:1]^T$

or as

$$\boldsymbol{x}_1 \times \boldsymbol{x}_2 = \begin{bmatrix} 0 & -w_1 & y_1 \\ w_1 & 0 & -x_1 \\ -y_1 & x_1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ w_2 \end{bmatrix} = \boldsymbol{T}\boldsymbol{x}_2$$

Please, note that homogeneous coordinates are used.

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Theorem

Let two points $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ be given in the projective space. Then the coefficients of the $\boldsymbol{p}$ line, which is defined by those two points, are determined as the cross product of their homogeneous coordinates

$$\boldsymbol{p} = \boldsymbol{x}_1 \times \boldsymbol{x}_2 = [a, b : c]^T$$

Proof

Let the line P^2 be defined in homogeneous coordinates as

$$ax + by + cw = 0$$

We are actually looking for a solution to the following equations:

$$\boldsymbol{p}^T \boldsymbol{x}_1 = 0 \qquad \boldsymbol{p}^T \boldsymbol{x}_2 = 0$$

where: $\boldsymbol{p} = [a, b : c]^T$

Note that c represents a "distance" from the origin of the coordinate system.

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It means that any point $\boldsymbol{x}$ that lies on the *p* line must satisfy both the equation above and the equation $\boldsymbol{p}^{\boldsymbol{T}}\boldsymbol{x} = 0$ in other words the $\boldsymbol{p}$ vector is defined as

$$\boldsymbol{p} = \boldsymbol{x}_1 \times \boldsymbol{x}_2 = det \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x_1 & y_1 & w_1 \\ x_2 & y_2 & w_2 \end{bmatrix}$$

We can write

$$(\boldsymbol{x}_1 \times \boldsymbol{x}_2)^{\boldsymbol{T}}\boldsymbol{x} = 0 \qquad \text{i.e.} \quad det \begin{bmatrix} x & y & w \\ x_1 & y_1 & w_1 \\ x_2 & y_2 & w_2 \end{bmatrix} = 0$$

Note that the Cross product and the Dot product is an instruction in Cg/HLSL on GPU.

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Evaluating the determinant $det\begin{bmatrix} a & b & c \\ x_1 & y_1 & w_1 \\ x_2 & y_2 & w_2 \end{bmatrix} = 0$

we get the line coefficients of the line *p* as:

$$a = det\begin{bmatrix} y_1 & w_1 \\ y_2 & w_2 \end{bmatrix} \qquad b = -det\begin{bmatrix} x_1 & w_1 \\ x_2 & w_2 \end{bmatrix} \qquad c = det\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$$

Note:

1. A line $ax + by + c = 0$ is a one parametric set of coefficients
   $$\boldsymbol{p} = [a, b : c]^T$$
   From two values $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ we have to compute 3 values, coefficients a , b and c

2. For w = 1 we get the standard cross product formula and the cross product defines the *p* line, i.e. $\boldsymbol{p} = \boldsymbol{x}_1 \times \boldsymbol{x}_2$ where:
   $$\boldsymbol{p} = [a, b : c]^T$$

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We have seen that computation of

- an intersection of two lines is given as $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$
- a line given by two points is given as $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$

Different scheme BUT

Those problems are DUAL.

Why algorithms are different??

Cross product is equivalent to a solution of a linear system of equations!

No division operations!

DUALITY APPLICATION

In the projective space P^2 points and lines are dual. Due to duality we can directly intersection of two lines as

$$\boldsymbol{x} = \boldsymbol{p}_1 \times \boldsymbol{p}_2 = det \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} = [x, y: w]^T$$

If the lines are parallel or close to parallel, the homogeneous coordinate $w \to 0$ and users have to take a decision – so there is no sequence in the code like $\boldsymbol{if}\ abs(\det(..)) \leq eps\ \boldsymbol{then}$ …in the procedure.

Generally computation can continue even if $w \to 0$ if projective space is used.

DISTANCE

Geometry is strongly connected with distances and their measurement, geometry education is strictly sticked to the Euclidean geometry, where the distance is measured as

$$d = \sqrt{(\Delta x)^2 + (\Delta y)^2} \quad \text{, resp.} \quad d = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2} \; .$$

This concept is convenient for a solution of basic geometric problems, but in many cases it results into quite complicated formula and there is a severe question of stability and robustness in many cases.

The main objection against the projective representation is that there is no metric.

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The distance of two points can be easily computed as

$$dist = \sqrt{\xi^2 + \eta^2}/(w_1 w_2)$$

where: $\xi = w_1 x_2 - w_2 x_1$ $\eta = w_1 y_2 - w_2 y_1$

Also a distance of a point $\boldsymbol{x}_0$ from a line in E^2 can be computed as

$$dist = \frac{\boldsymbol{a}^T \boldsymbol{x}_0}{w_0 \sqrt{a^2 + b^2}}$$

where: $\boldsymbol{x_0} = [x_0, y_0 : w_0]^T$ $\boldsymbol{a} = [a, b : c]^T$

The extension to E^3/P^3 is simple and the distance of a point $\boldsymbol{x}_0$ from a plane in E^3 can be computed as

$$dist = \frac{\boldsymbol{a}^T \boldsymbol{x}_0}{w_0 \sqrt{a^2 + b^2 + c^2}}$$

where: $\boldsymbol{x_0} = [x_0, y_0, z_0 : w_0]^T$ $\boldsymbol{a} = [a, b, c : d]^T$.

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In many cases we do not need actually a *distance*, e.g. for a decision which object is closer, and *distance*^{2} can be used instead, i.e. for the E^2 case

$$dist^2 = \frac{(\boldsymbol{a}^T \boldsymbol{x}_0)^2}{{w_0}^2(a^2+b^2)} = \frac{(\boldsymbol{a}^T \boldsymbol{x}_0)^2}{{w_0}^2\, \boldsymbol{n}^T \boldsymbol{n}}$$

where: $\boldsymbol{a} = \lfloor a, b : c \rfloor^T = \lfloor \boldsymbol{n} : c \rfloor^T$ and the normal vector $\boldsymbol{n}$ is not normalized

If we are comparing distances of points from the given line *p* we can use "*pseudo-distance*" for comparisons

$$(pseudo\_dist)^2 = \frac{(\boldsymbol{a}^T \boldsymbol{x}_0)^2}{{w_0}^2}$$

Similarly for a plane ρ in the case of E^3

$$dist^2 = \frac{(\boldsymbol{a}^T \boldsymbol{x}_0)^2}{{w_0}^2(a^2+b^2+c^2)} = \frac{(\boldsymbol{a}^T \boldsymbol{x}_0)^2}{{w_0}^2\, \boldsymbol{n}^T \boldsymbol{n}} \quad \text{and} \quad (pseudo\_dist)^2 = \frac{(\boldsymbol{a}^T \boldsymbol{x}_0)^2}{{w_0}^2}$$

where: $\boldsymbol{a} = \lfloor a, b, c : d \rfloor^T = \lfloor \boldsymbol{n} : d \rfloor^T$

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Computation in Projective Space

- Cross product definition
- A plane $\boldsymbol{\rho}$ is determined as a cross product of three given points

Due to the duality

- An intersection point $\boldsymbol{x}$ of three planes is determined as a cross product of three given planes.

$$\boldsymbol{x}_1 \times \boldsymbol{x}_2 \times \boldsymbol{x}_3 = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} & \boldsymbol{l} \\ x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \\ x_3 & y_3 & z_3 & w_3 \end{vmatrix}$$

$$\boldsymbol{\rho}_1 \times \boldsymbol{\rho}_2 \times \boldsymbol{\rho}_3 = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} & \boldsymbol{l} \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix}$$

Computation of generalized cross product is equivalent to a solution of a linear system of equations
=> no division operation!

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Geometric transformations with points

(note $X = {}^{x}/_{w}$, $Y = {}^{y}/_{w}$, $w \neq 0$):

Translation by a vector $(A, B) \triangleq [a, b: c]^T$, i.e. $A = a/c$, $B = b/c$, $c \neq 0$:

In the Euclidean space: $\boldsymbol{x}' = \boldsymbol{T}\boldsymbol{x}$

$$\begin{bmatrix} x' \\ y' \\ w \end{bmatrix} = \begin{bmatrix} 1 & 0 & A \\ 0 & 1 & B \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} x + Aw \\ y + Bw \\ w \end{bmatrix} \triangleq \begin{bmatrix} x/w + A \\ y/w + B \\ 1 \end{bmatrix} = \begin{bmatrix} X + A \\ Y + A \\ 1 \end{bmatrix}$$

In the projective space: $\boldsymbol{x}' = \boldsymbol{T}'\boldsymbol{x}$

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} c & 0 & a \\ 0 & c & b \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} cx + aw \\ cy + bw \\ cw \end{bmatrix} \triangleq \begin{bmatrix} (cx + aw)/(cw) \\ (cy + bw)/(cw) \\ 1 \end{bmatrix} = \begin{bmatrix} {}^{x}/_{w} + {}^{a}/_{c} \\ {}^{y}/_{w} + {}^{b}/_{c} \\ 1 \end{bmatrix} = \begin{bmatrix} X + A \\ Y + B \\ 1 \end{bmatrix}$$

and $\det(\boldsymbol{T}') = c^3$. For $c = 1$ we get a standard formula

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & A \\ 0 & 1 & B \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

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Rotation by an angle $(cos\varphi, sin\varphi) = \left(\frac{a}{c}, \frac{b}{c}\right) \triangleq [a, b : c]^T$

In the Euclidean space: $\boldsymbol{x}' = \boldsymbol{R}\boldsymbol{x}$

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} cos\varphi & -sin\varphi & 0 \\ sin\varphi & cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} \triangleq \begin{bmatrix} cos\varphi & -sin\varphi & 0 \\ sin\varphi & cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

In the projective space: $\boldsymbol{x}' = \boldsymbol{R}'\boldsymbol{x}$

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & -b & 0 \\ b & a & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} ax - by \\ bx + ay \\ cw \end{bmatrix} \triangleq$$

$$\begin{bmatrix} (ax - by)/(cw) \\ (bx + ay)/(cw) \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{x}{w}\frac{a}{c} - \frac{y}{w}\frac{b}{c} \\ \frac{x}{w}\frac{b}{c} + \frac{y}{w}\frac{a}{c} \\ 1 \end{bmatrix} = \begin{bmatrix} Xcos\varphi - Ysin\varphi \\ Xsin\varphi + Ycos\varphi \\ 1 \end{bmatrix}$$

as $c^2 = (a^2 + b^2)$ by definition, $\det(\boldsymbol{R}') = (a^2 + b^2)c = c^3$

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Scaling by a factor $(S_x, S_y) = (\frac{s_x}{w_s}, \frac{s_y}{w_s}) \triangleq \left[s_x, s_y : w_s\right]^T$

$$\boldsymbol{x}' = \boldsymbol{S}\boldsymbol{x} \qquad \begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

$$\boldsymbol{x}' = \boldsymbol{S}'\boldsymbol{x} \qquad \begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & w_s \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

$$\det(\boldsymbol{S}') = s_x\, s_y\, w_s$$

It is necessary to note that the determinant of a transformation matrix $\boldsymbol{Q}$, i.e. matrices $\boldsymbol{T}', \boldsymbol{R}', \boldsymbol{S}'$, is $\det(\boldsymbol{Q}) \neq 1$ in general, but as the formulation is in the projective space, there is no need to “normalize” transformations to $\det(\boldsymbol{Q}) = 1$ even for rotation.

It can be seen that if the parameters of a geometric transformation are given in the homogeneous coordinates, no division operation is needed at all.

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Transformation of lines and planes

	E^2	E^3
	$\boldsymbol{p} = \boldsymbol{x}_1 \times \boldsymbol{x}_2$	$\boldsymbol{\rho} = \boldsymbol{x}_1 \times \boldsymbol{x}_2 \times \boldsymbol{x}_3$
Dual problem	$\boldsymbol{x} = \boldsymbol{p}_1 \times \boldsymbol{p}_2$	$\boldsymbol{x} = \boldsymbol{\rho}_1 \times \boldsymbol{\rho}_2 \times \boldsymbol{\rho}_3$

In graphical applications position of points are changed by an interaction, i.e. $\boldsymbol{x}' = \boldsymbol{T}\boldsymbol{x}$.

The question is how coefficients of a line, resp. a plane are changed without need to recompute them from the definition.

It can be proved that

$$\boldsymbol{p}' = (\boldsymbol{T}\boldsymbol{x}_1) \times (\boldsymbol{T}\boldsymbol{x}_2) = det(\boldsymbol{T})(\boldsymbol{T}^{-1})^T\boldsymbol{p} \triangleq (\boldsymbol{T}^{-1})^T\boldsymbol{p}$$

or

$$\boldsymbol{\rho}' = (\boldsymbol{T}\boldsymbol{x}_1) \times (\boldsymbol{T}\boldsymbol{x}_2) \times (\boldsymbol{T}\boldsymbol{x}_3) = det(\boldsymbol{T})(\boldsymbol{T}^{-1})^T\boldsymbol{\rho} \triangleq (\boldsymbol{T}^{-1})^T\boldsymbol{\rho}$$

Transformation of lines and planes

As the computation is made **in the projective space** we can write

$$\boldsymbol{p}' = (\boldsymbol{T}^{-1})^T \boldsymbol{p} = [a',\ b' : c']^T \qquad \text{for lines in } E^2$$

or

$$\boldsymbol{\rho}' = (\boldsymbol{T}^{-1})^T \boldsymbol{\rho} = [a',\ b', c' : d']^T \qquad \text{for planes in } E^3$$

THIS SIMPLIFIES COMPUTATIONS

Transformation matrices for lines, resp. for planes are **DIFFERENT** from transformations for points! Note that a normal vector of a line is actually a co-vector, i.e. an oriented "surface".

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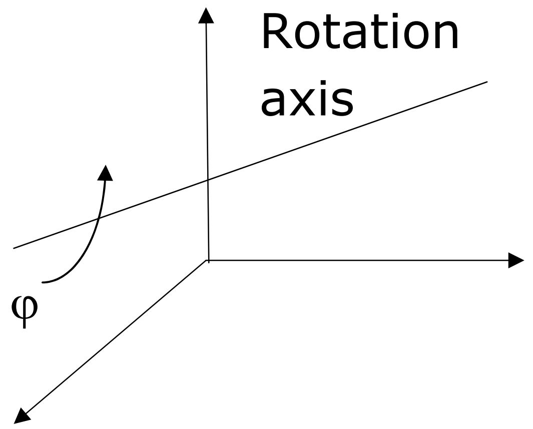
Transformation of lines and planes

Transformation about a general axis in E^3 / P^3

$$\boldsymbol{x}(t) = \boldsymbol{x}_A + \boldsymbol{s}t$$



Usually used transformation ($\boldsymbol{T}$ is translation):

$$\boldsymbol{Q} = \boldsymbol{T}^{-1}\boldsymbol{R}_{zx}^{-1}\boldsymbol{R}_{yz}^{-1}\boldsymbol{R}(\varphi)\boldsymbol{R}_{zx}\boldsymbol{R}_{xy}\boldsymbol{T}$$

$$\boldsymbol{R}_{yz} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{c}{\sqrt{b^2+c^2}} & \dfrac{-b}{\sqrt{b^2+c^2}} & 0 \\ 0 & \dfrac{b}{\sqrt{b^2+c^2}} & \dfrac{c}{\sqrt{b^2+c^2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\boldsymbol{s} = [a, b, c]^T$ is an axis directional vector. This is unstable if $\sqrt{b^2+c^2} \to 0$ and not precise if $b^2 \gg c^2$ or vice versa.

That is generally computationally complex and unstable as *a user has to select which axis is to be used for a rotation*

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Transformation of lines and planes

Transformation about an axis $\boldsymbol{n}$
in the Euclidean space E^3

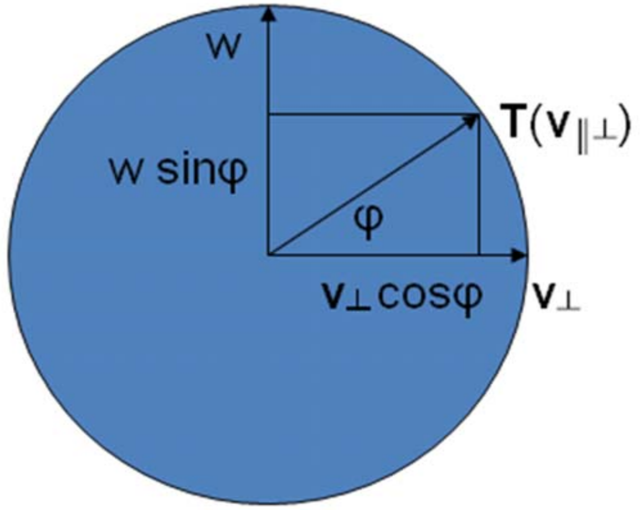
$$\boldsymbol{X} = \boldsymbol{X}\, cos\varphi + (1 - cos\varphi)(\,\boldsymbol{n}^T\boldsymbol{X}).\boldsymbol{n} + (\boldsymbol{n} \times \boldsymbol{X})sin\varphi$$

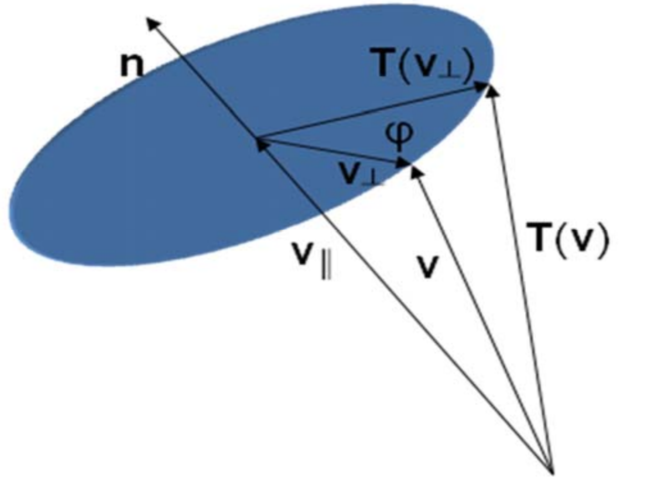
$$\boldsymbol{Q} = \boldsymbol{I}cos\varphi + (1 - cos\varphi)(\boldsymbol{n}\otimes\boldsymbol{n}) + \boldsymbol{W}sin\varphi$$

where: $\boldsymbol{n}\otimes\boldsymbol{n} = \boldsymbol{n}\,.\,\boldsymbol{n}^T$ is a matrix.
In the Euclidean space E^3 the vector $\boldsymbol{n}$ has to be normalized

The matrix $\boldsymbol{W}$ is defined as: $\boldsymbol{Wv} = \boldsymbol{w} \times \boldsymbol{v}$

$$\boldsymbol{W} = \begin{bmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_y & w_x & 0 \end{bmatrix}$$ in our case





Projection

Standard projection

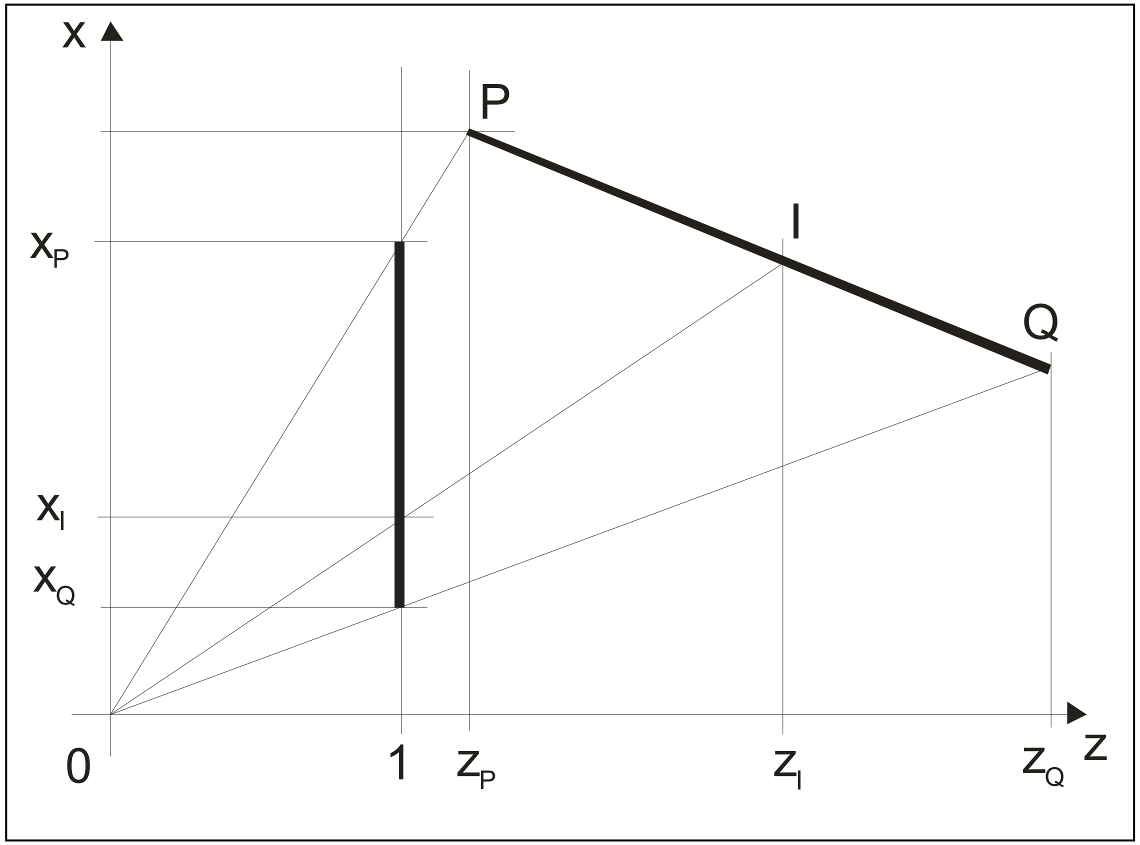
$$\begin{bmatrix} x_I \\ y_I \\ w_I \end{bmatrix} = \boldsymbol{T}_{projection} \begin{bmatrix} x_I \\ y_I \\ 1 \end{bmatrix}$$

How to determine z_I coordinate?

$$x_I = (1-\lambda)x_P + \lambda x_Q$$

$$y_I = (1-\lambda)y_P + \lambda y_Q$$

$$\frac{1}{z_I} = (1-\lambda)\frac{1}{z_P} + \lambda\frac{1}{z_Q}$$



⇨ “Reverse” depth computation, e.g. for correct intensity computation if perspective projection is used.

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Computation in Projective Space

Linear parametrization:

$\boldsymbol{X}(t) = \boldsymbol{X}_0 + (\boldsymbol{X}_1 - \boldsymbol{X}_0)\, t \qquad t \in (-\infty, \infty)$

Non-linear monotonous parametrization:

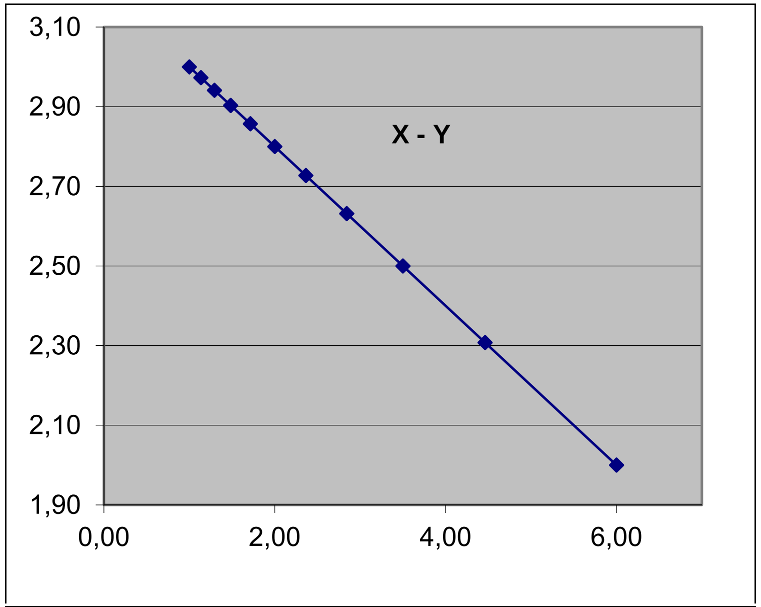
$\boldsymbol{x}(t) = \boldsymbol{x}_0 + (\boldsymbol{x}_1 - \boldsymbol{x}_0)\, t \qquad t \in (-\infty, \infty)$

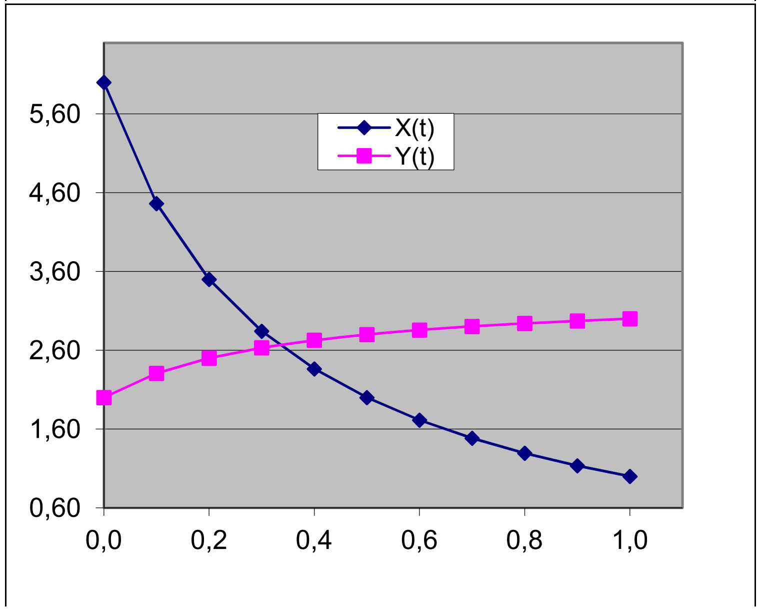
$x(t) = x_0 + (x_1 - x_0)\, t \qquad y(t) = y_0 + (y_1 - y_0)\, t$

$z(t) = z_0 + (z_1 - z_0)\, t \qquad w(t) = w_0 + (w_1 - w_0)\, t$

- it means that we can interpolate using homogeneous coordinates without a need of "normalization" to E^k !!
- Homogeneous coordinate $w \geq 0$

In many algorithms, we need "monotonous" parameterization, only





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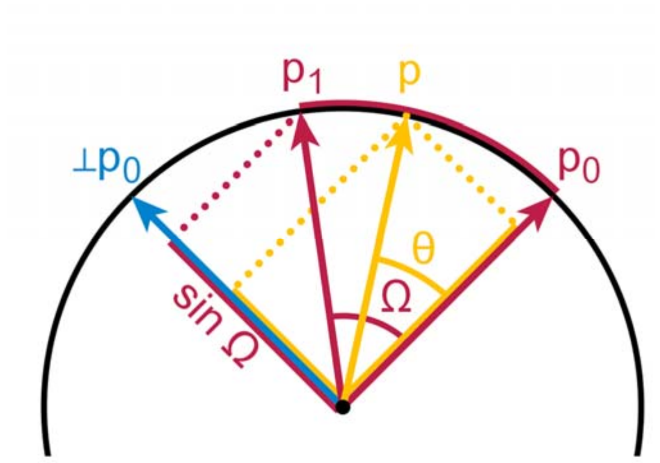
Computation in Projective Space



Courtesy of wikipedia

Spherical interpolation

$$slerp(\boldsymbol{X}_0, \boldsymbol{X}_1, t) = \frac{\sin[(1-t)\Omega]}{\sin\Omega}\boldsymbol{X}_0 + \frac{\sin[t\Omega]}{\sin\Omega}\boldsymbol{X}_1$$

Instability occurs if $\Omega \rightarrow k\pi$.

Mathematically formula is correct;

in practice the **code is generally incorrect**! $[\ \frac{0}{0}\ ]$

$$slerp_p(\boldsymbol{X}_0, \boldsymbol{X}_1, t) = \left[\frac{\sin[(1-t)\Omega]\boldsymbol{X}_0 + \sin[t\Omega]\boldsymbol{X}_1}{\sin\Omega}\right]$$

$$\equiv [\sin[(1-t)\Omega]\boldsymbol{X}_0 + \sin[t\Omega]\boldsymbol{X}_1 : \sin\Omega]^T$$

projective scalar use

Homogeneous coordinate

Homogeneous coordinates
=> better numerical stability &

division operation can be postponed

Computation in Projective Space

$$\boldsymbol{x}(t) = slerp(\boldsymbol{x}_0, \boldsymbol{x}_1, t) = \frac{\sin[(1-t)\Omega]}{\sin\Omega}\boldsymbol{x}_0 + \frac{\sin[t\Omega]}{\sin\Omega}\boldsymbol{x}_1$$
$$\triangleq [\sin[(1-t)\Omega]\,\boldsymbol{x}_0 + \sin[t\Omega]\,\boldsymbol{x}_1 : \sin\Omega]^{\mathrm{T}}$$

What is a result in the Euclidean space of a spherical interpolation with non-linear parameterization, i.e. $w_i \neq 1$?

$$x(t) = \frac{\sin[(1-t)\Omega]}{\sin\Omega}x_0 + \frac{\sin[t\Omega]}{\sin\Omega}x_1 \qquad w(t) = \frac{\sin[(1-t)\Omega]}{\sin\Omega}w_0 + \frac{\sin[t\Omega]}{\sin\Omega}w_1$$

$$\boldsymbol{X}(t) = \frac{\frac{\sin[(1-t)\Omega]}{\sin\Omega}x_0 + \frac{\sin[t\Omega]}{\sin\Omega}x_1}{\frac{\sin[(1-t)\Omega]}{\sin\Omega}w_0 + \frac{\sin[t\Omega]}{\sin\Omega}w_1} = \frac{\sin[(1-t)\Omega]\,x_0 + \sin[t\Omega]\,x_1}{\sin[(1-t)\Omega]\,w_0 + \sin[t\Omega]\,w_1}$$

If $w_i \neq 1$

If represented as "projective scalar" value

$$\boldsymbol{x}(t) = [\sin[(1-t)\Omega]\,x_0 + \sin[t\Omega]\,x_1 \; : \; \sin[(1-t)\Omega]\,w_0 + \sin[t\Omega]\,w_1]^T$$

=> Better numerical stability

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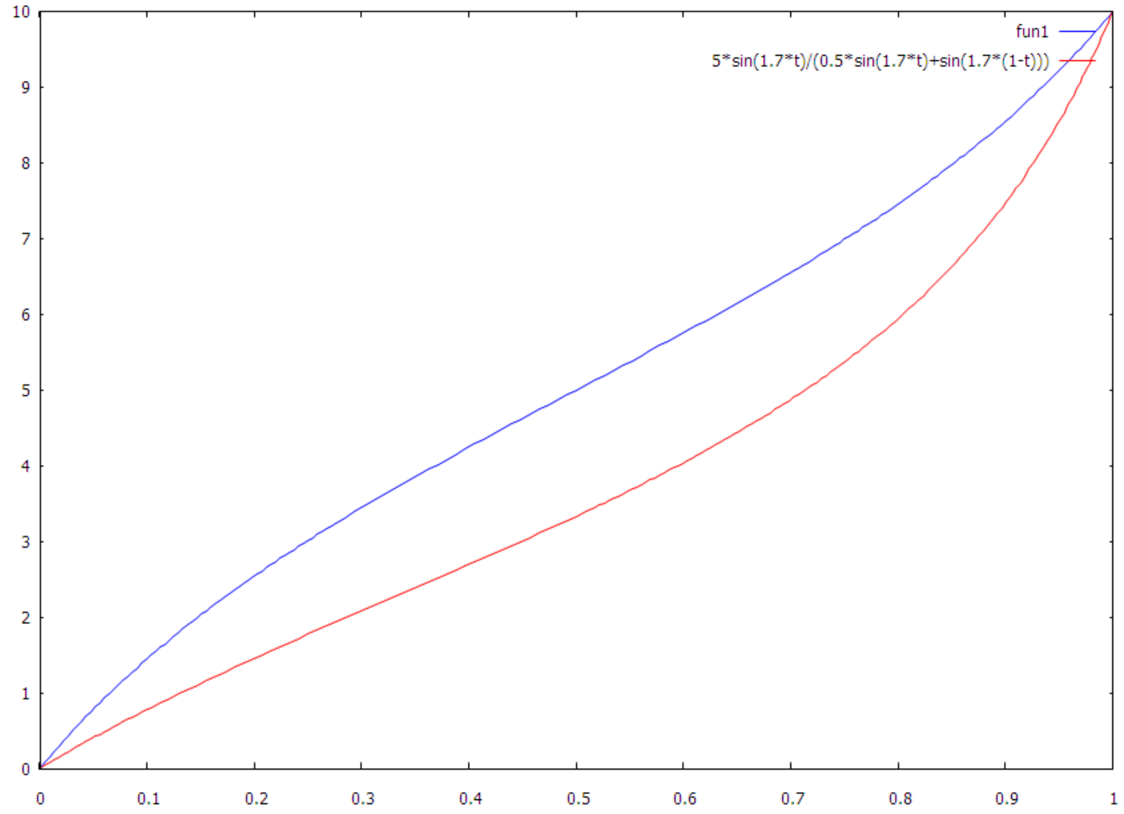
Influence of homogeneous values $w_i \neq 1$ & $w_i \neq 0$ (Fun1 - $w_i = 1$)

Can be used for:

- relative position comparison
- interpolation if a “linear” parameterization is required $w_i = 1$



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Computation in Projective Space

Intersection line – plane (Cyrus-Beck clipping algorithm)

$\boldsymbol{x}(t) = \boldsymbol{x}_0 + (\boldsymbol{x}_1 - \boldsymbol{x}_0)\, t = \boldsymbol{x}_0 + \boldsymbol{s}\, t$ Computation in projective space !

$\boldsymbol{s} = \boldsymbol{x}_1 - \boldsymbol{x}_0 = [x_1 - x_0, y_1 - y_0, z_1 - z_0 : \; w_1 - w_0]^T$ $t \in (-\infty, \infty)$

$\boldsymbol{a}^T\boldsymbol{x} = 0$ $ax + by + cz + dw = 0$ $\boldsymbol{a} = [a, b, c : d]^T$

Euclidean solution:

$$t = -\frac{\boldsymbol{a}^T\boldsymbol{x}_0}{\boldsymbol{a}^T\boldsymbol{s}}$$

Projective solution: $\tau = -\boldsymbol{a}^T\boldsymbol{x}_0$ and $\tau_w = \boldsymbol{a}^T\boldsymbol{s}$ $\boldsymbol{t} = [\tau, \tau_w]^T$

if $\tau_w < 0$ **then** $\boldsymbol{t} \coloneqq -\boldsymbol{t}$

Test: **if** $t < t_{min}$ **then** ...modification:

if $\tau * \tau_{min_w} > \tau_w * \tau_{min}$ **then** .....

- An intersection of a plane with a line in E^2 / E^3 can be computed efficiently, **no division operation**
- Comparison operations must be modified!!!
- => Cyrus-Beck line clipping algorithm 10-25% faster

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Line Clipping – convex polygon

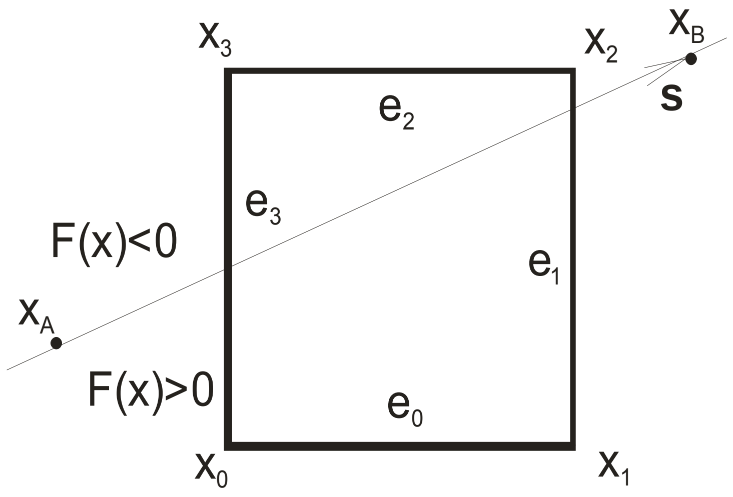
```
procedure CLIP_Line ( $\boldsymbol{x}_A$ , $\boldsymbol{x}_B$ );
/* $\boldsymbol{x}_A=[x_A,y_A:w_A]^T$ $\boldsymbol{x}_B=[x_B,y_B:w_B]^T$ */
begin /* $\boldsymbol{p}=[a,b:c]^T$ given - NO STEP 1 */
{1}$\boldsymbol{p} := \boldsymbol{x}_A \times \boldsymbol{x}_B$;     /* $\boldsymbol{p}$: $ax+by+c = 0$ */
{2}for k:=0 to N-1 do /*$\boldsymbol{x}_k=[x_k,y_k,w_k]^T$*/
{3}      if $\boldsymbol{p}^T\boldsymbol{x}_k \geq 0$ then $\boldsymbol{c}_k:=1$ else $c_k:=0$;
{4}if $\boldsymbol{c}=[0…0]^T$ or $\boldsymbol{c}=[1…1]^T$ then EXIT;
{5}i:= TAB1[c];    j:= TAB2[c];
{6}$\boldsymbol{x}_A := \boldsymbol{p} \times \boldsymbol{e}_i$ ;    $\boldsymbol{x}_B := \boldsymbol{p} \times \boldsymbol{e}_j$ ;
{7}DRAW ($\boldsymbol{x}_A$; $\boldsymbol{x}_B$ )    * $\boldsymbol{e}_i$ – i-th edge */
end /* CLIP_Line */
/* c identifies an edge intersected */
```

TOO COMPLEX?

NO SIMPLE, ROBUST and FAST

Line clipping algorithms in E^2

- Cohen-Sutherland
- Liang-Barsky
- Hodgman
- Skala – modification of Clip_L for line segments



- Skala,V.: A new approach to line and line segment clipping in homogeneous coordinates, The Visual Computer, SpringerVol.21, No.11, pp.905-914, 2005

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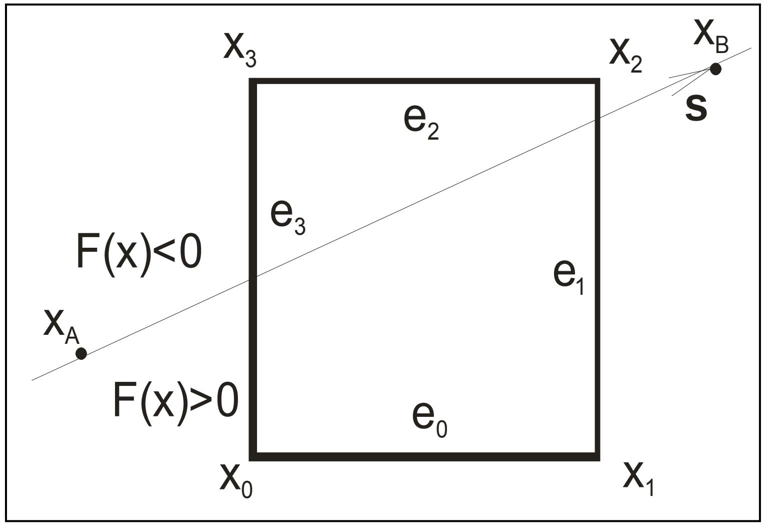
For a rectangular normalized window

$\boldsymbol{x}_A := \boldsymbol{p} \times \boldsymbol{e}_i$

For the edge $y = -1$, i.e. $y + 1 = 0$



$$\boldsymbol{x}_A = [x_A, y_A : w_A]^T = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a & b & c \\ 0 & 1 & 1 \end{vmatrix} = [b - c, -a : a]$$

$$\triangleq [\frac{b-c}{a}, \frac{-a}{a} : 1]^T = [\frac{b-c}{a}, -1 : 1]^T$$

Actually expression for y_A , resp. for x_A is given by the window edge.

No multiplication or division operations

A simple modification if a line is given
(in the Euclidean or projective space) as $\boldsymbol{x}(t) = \boldsymbol{x}_A + \boldsymbol{s}t$

Simple modification for non-convex polygon but it requires intersections sorting => $O(M\ lgM)$

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/* Additional code for Line Segment clipping colored */

```
function CODE (x); /* Cohen Sutherland window coding */
begin  c:= [0000];     /* land / lor – bitwise  operations and / or */
    if $x<x_{min}$ then c:=[1000] else if $x>x_{max}$ then c:= [0100];
    if $y<y_{min}$ then c:=c lor [1001] else if $y>y_{max}$ then c:=c lor [0010];
    CODE := c
end /*CODE */;
procedure Clip_LineSegment ($x_A$ , $x_B$); /* short x long line segments */
begin  $c_A$ := CODE ($x_A$);    $c_B$ := CODE ($x_B$);
    if ($c_A$ lor $c_B$) = 0 then { output ($x_A$; $x_B$ ); EXIT }      /*  inside */
    if ($c_A$ land $c_B$) $\neq$ 0 then EXIT;                          /* outside */
    p := $x_A$ x $x_B$;                        /* $ax+by+c = 0$; $p = [a,b:c]^T$ */
    for k:=0 to 3 do                       /* $x_k=[x_k,y_k: w_k]^T$   $c=[ c_3, c_2, c_1, c_0 ]^T$ */
        if $p^T x_k \geq 0$ then $c_k$:=1 else $c_k$:=0;
    if c = $[0000]^T$ or c = $[1111]^T$ then EXIT;
    i:= TAB1[c];    j:= TAB2[c];
    /* original code $x_A$ :=  $p$ x $e_i$ ;  $x_B$ :=  $p$ x $e_j$ ; DRAW ($x_A$; $x_B$ ) */
```

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```
    /* additional code for Line Segment clipping */
    if $c_A \neq 0$ and $c_B \neq 0$
    then { $x_A := p \times e_i$ ;  $x_B := p \times e_j$ } /* two intersections */
    else /* only one intersection point */
      if $c_A = 0$ then /* $x_B$ is outside */
    { if $c_B$ land MASK[c] $\neq 0$
              then  $x_B := p \times e_i$
              else  $x_B := p \times e_j$
    }
      else if $c_B = 0$ then /* $x_A$ is outside */
        { if $c_A$ land MASK[c] $\neq 0$
              then  $x_A := p \times e_i$
              else  $x_A := p \times e_j$
        };
    output ($x_A$ , $x_B$ )
end /* Clip_LineSegment */
```

c	c_3	c_2	c_1	c_0	TAB1	TAB2	MASK
0	0	0	0	0	None	None	None
1	0	0	0	1	0	3	0100
2	0	0	1	0	0	1	0100
3	0	0	1	1	1	3	0010
4	0	1	0	0	1	2	0010
5	0	1	0	1	N/A	N/A	N / A
6	0	1	1	0	0	2	0100
7	0	1	1	1	2	3	1000
8	1	0	0	0	2	3	1000
9	1	0	0	1	0	2	0100
10	1	0	1	0	N/A	N/A	N / A
11	1	0	1	1	1	2	0010
12	1	1	0	0	1	3	0010
13	1	1	0	1	0	1	0100
14	1	1	1	0	0	3	0100
15	1	1	1	1	None	None	None

Algorithm can be extended to convex polygon clipping & modified for parametric lines as well

For a convex n-sided polygon the table can be generated synthetically.

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Computation in Projective Space - Barycentric coordinates

Let us consider a triangle with vertices $\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_{31}$

A position of any point $\boldsymbol{X} \in E^2$ can be expressed as

$$a_1 X_1 + a_2 X_2 + a_3 X_3 = X$$

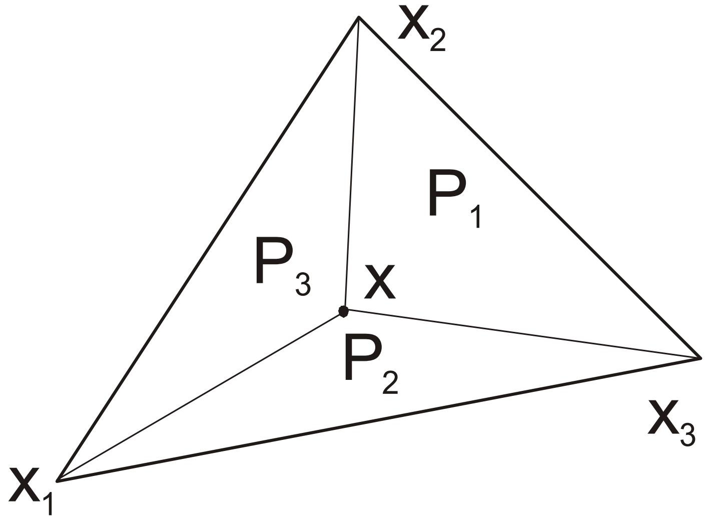
$$a_1 Y_1 + a_2 Y_2 + a_3 Y_3 = Y$$

Additional condition:

$$a_1 + a_2 + a_3 = 1$$

$$0 \le a_i \le 1 \qquad i = 1, \dots, 3$$

$$a_i = \frac{P_i}{P}$$



Linear system has to be solved

If points $\boldsymbol{x}_i$ are given as $\boldsymbol{x}_i = [x_i, y_i, z_i : w_i]^T$ and $w_i \neq 1$ then $\boldsymbol{x}_i$ have to be "normalized" to $w_i = 1$, i.e. 4 * 3 = 12 division operations are used.

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Computation in Projective Space

$$b_1X_1 + b_2X_2 + b_3X_3 + b_4X = 0$$

$$b_1Y_1 + b_2Y_2 + b_3Y_3 + b_4Y = 0$$

$$b_1 + b_2 + b_3 + b_4 = 1 \qquad b_i = -a_ib_4 \qquad b_4 \neq 0 \qquad i = 1, \dots, 3$$

Rewriting

$$\begin{bmatrix} X_1 & X_2 & X_3 & X \\ Y_1 & Y_2 & Y_3 & Y \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \mathbf{0}$$

$$\boldsymbol{b} = [b_1, b_2, b_3, b_4]^T$$
$$\boldsymbol{\xi} = [X_1, X_2, X_3, X]^T$$
$$\boldsymbol{\eta} = [Y_1, Y_2, Y_3, Y]^T$$
$$\boldsymbol{\omega} = [1, 1, 1,1]^T$$

A solution of the linear system of equations (LSE) is equivalent to generalized cross product:

$$\boldsymbol{b} = \boldsymbol{\xi} \times \boldsymbol{\eta} \times \boldsymbol{\omega}$$

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Computation in Projective Space

For $w_i \neq 0$ each column can be multiplied by the relevant w_i

$$\begin{bmatrix} w_1X_1 & w_2X_2 & w_3X_3 & wX \\ w_1Y_1 & w_2Y_2 & w_3Y_3 & wY \\ w_1 & w_2 & w_3 & w \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \mathbf{0} \qquad \begin{bmatrix} x_1 & x_2 & x_3 & x \\ y_1 & y_2 & y_3 & y \\ w_1 & w_2 & w_3 & w \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \mathbf{0}$$

where again

$$\boldsymbol{b} = [b_1, b_2, b_3, b_4]^T \quad \boldsymbol{\xi} = [x_1, x_2, x_3, x]^T \quad \boldsymbol{\eta} = [y_1, y_2, y_3, y]^T \quad \boldsymbol{\omega} = [w_1, w_2, w_3, w]^T$$

Barycentric coordinates (Euclidean):

$$0 \leq (-b_1 : w_2, w_3, w) \leq 1 \qquad 0 \leq (-b_2 : w_3, w_1, w) \leq 1 \qquad 0 \leq (-b_3 : w_1, w_2, w) \leq 1$$

=> Tests – point in triangle, point in tetrahedron -

all in homogeneous coordinates

=> **new entities**: *projective scalar, projective vector*

(Skala,V.: Barycentric coordinates computation in homogeneous coordinates, Computers&Graphics, 2008)

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Computation in Projective Space

Area of a triangle

$$P = \frac{1}{2}\boldsymbol{x}_1^T.(\boldsymbol{x}_2 \times \boldsymbol{x}_3)/(w_1 w_2 w_3)$$

Volume of a tetrahedron

$$V = \frac{1}{6}\boldsymbol{x}_1^T.(\boldsymbol{x}_2 \times \boldsymbol{x}_3 \times \boldsymbol{x}_4)/(w_1 w_2 w_3 w_4)$$

As the principle of duality is valid, one could ask: *What is a “dual” value G to a computation of the area P if the triangle is given by three lines in the “normalized” form, e.g. $\boldsymbol{a}_1^T.(\boldsymbol{a}_2 \times \boldsymbol{a}_3)$ instead of three points?*

$$G = \boldsymbol{a}_1^T.(\boldsymbol{a}_2 \times \boldsymbol{a}_3) \triangleq \begin{vmatrix} cos\alpha_1 & cos\alpha_2 & cos\alpha_3 \\ sin\alpha_1 & sin\alpha_2 & sin\alpha_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = \begin{vmatrix} 1 & cos\alpha_2 & cos\alpha_3 \\ 0 & sin\alpha_2 & sin\alpha_3 \\ 0 & 0 & d_3 \end{vmatrix} = d_3 sin\alpha_2$$

$$= d_3.a/(2R) = P/R$$

It can be seen that $G = d_3 sin\alpha_2 = P/R$, where: a is the length of the line segment on $\boldsymbol{a}_3$ and R is a radius of the circumscribing circle.

=> value G can be used as criterion for a quality triangular meshes.

- Skala,V.: Geometric Computation, Duality and Projective Space, IW-LGK workshop proceedings, ISBN 978-3-86780-244-4, pp.105-111, Dresden University of Technology, 2011

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Computation in Projective Space

Line in E^3 as Two Plane Intersection

Standard formula in the Euclidean space

$$\boldsymbol{\rho}_1 = [a_1, b_1, c_1 : d_1]^T = [\boldsymbol{n}_1^T : d_1]^T \qquad \boldsymbol{\rho}_2 = [a_2, b_2, c_2 : d_2]^T = [\boldsymbol{n}_2^T : d_2]^T$$

Line given as an intersection of two planes

$$\boldsymbol{s} = \boldsymbol{n}_1 \times \boldsymbol{n}_2 \equiv [a_3, b_3, c_3]^T \qquad \boldsymbol{x}(t) = \boldsymbol{x}_0 + \boldsymbol{s}t$$

$$x_0 = \frac{d_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} - d_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}}{DET} \qquad y_0 = \frac{d_2 \begin{vmatrix} a_3 & c_3 \\ a_1 & c_1 \end{vmatrix} - d_1 \begin{vmatrix} a_3 & c_3 \\ a_2 & c_2 \end{vmatrix}}{DET}$$

$$z_0 = \frac{d_2 \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} - d_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}}{DET} \qquad DET = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

The formula is quite “horrible” one and for students not acceptable as it is too complex and they do not see from the formula comes from.

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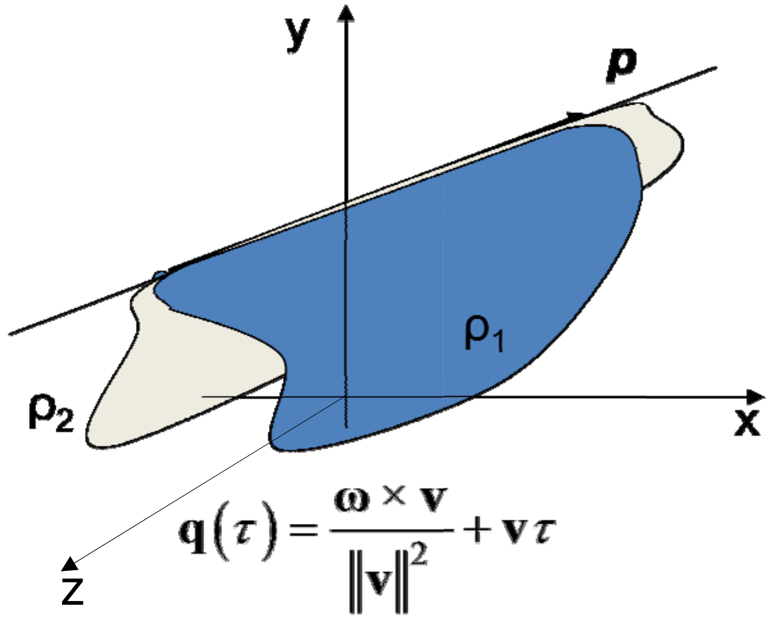
Computation in Projective Space

Line in E3 as Two Plane Intersection

- Standard formula in the Euclidean space
- Plücker's coordinates
  a line is given by two points. DUALITY – a point is dual to
  a plane and vice versa =>
  an intersection of two planes can be computed as a dual problem
  But computationally **expensive** computation
- Projective formulation and simple computation



$$\mathbf{q}(\tau) = \frac{\boldsymbol{\omega} \times \mathbf{v}}{\|\mathbf{v}\|^2} + \mathbf{v}\tau$$

$$\boldsymbol{\omega} = \begin{bmatrix} l_{41} & l_{42} & l_{43} \end{bmatrix}^T \quad \mathbf{v} = \begin{bmatrix} l_{23} & l_{31} & l_{12} \end{bmatrix}^T$$

$$\mathbf{L} = \mathbf{a}_0\mathbf{a}_1^T - \mathbf{a}_1\mathbf{a}_0^T \quad \textit{tensor product} - \textit{matrix}$$

$$\mathbf{a}_i = \begin{bmatrix} a_i, b_i, c_i, d_i \end{bmatrix}^T \quad i = 1,2$$

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Computation in Projective Space

Line in E3 as Two Plane Intersection

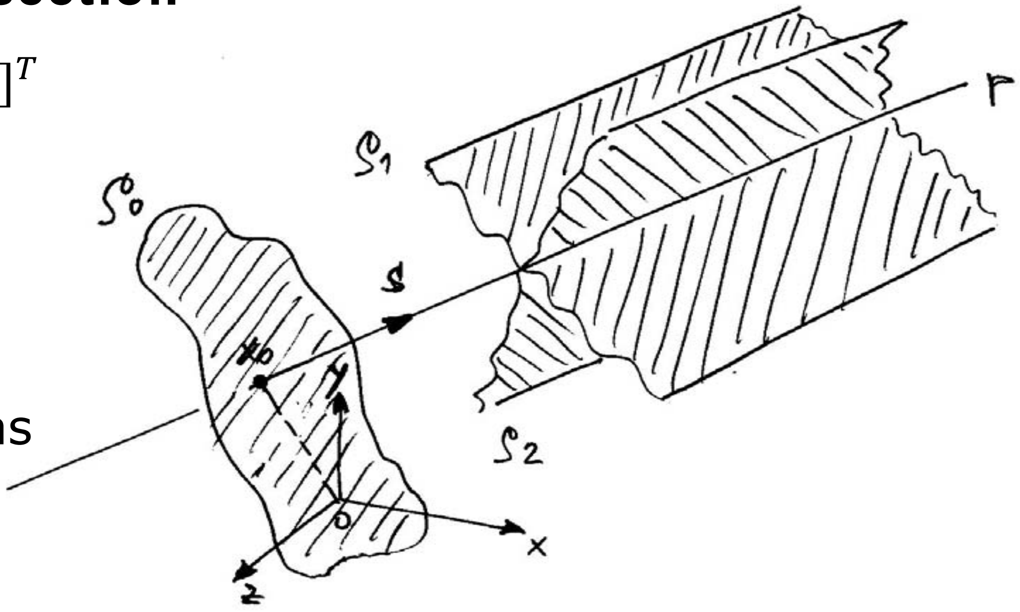
$\boldsymbol{\rho}_1 = [a_1, b_1, c_1 : d_1]^T$ $\boldsymbol{\rho}_2 = [a_2, b_2, c_2 : d_2]^T$

normal vectors are

$\boldsymbol{n}_1 = [a_1, b_1, c_1]^T$ $\boldsymbol{n}_2 = [a_2, b_2, c_2]^T$

Directional vector of a line
of two planes $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_1$ is given as

$$\boldsymbol{s} = \boldsymbol{n}_1 \times \boldsymbol{n}_2$$



"starting" point $\boldsymbol{x}_0$ **???**

A plane $\boldsymbol{\rho}_0$ passing the origin with a normal vector $\boldsymbol{s}$, $\boldsymbol{\rho}_0 = [a_0, b_0, c_0 : 0]^T$

The point $\boldsymbol{x}_0$ is defined as $\boldsymbol{x}_0 = \boldsymbol{\rho}_1 \times \boldsymbol{\rho}_2 \times \boldsymbol{\rho}_0$

Simple formula for matrix-vector architectures like GPU and parallel processing. *Compare the standard and projective formulas*

Computation in Projective Space – the nearest point

Find the nearest point on an intersection of two planes to the given point ξ

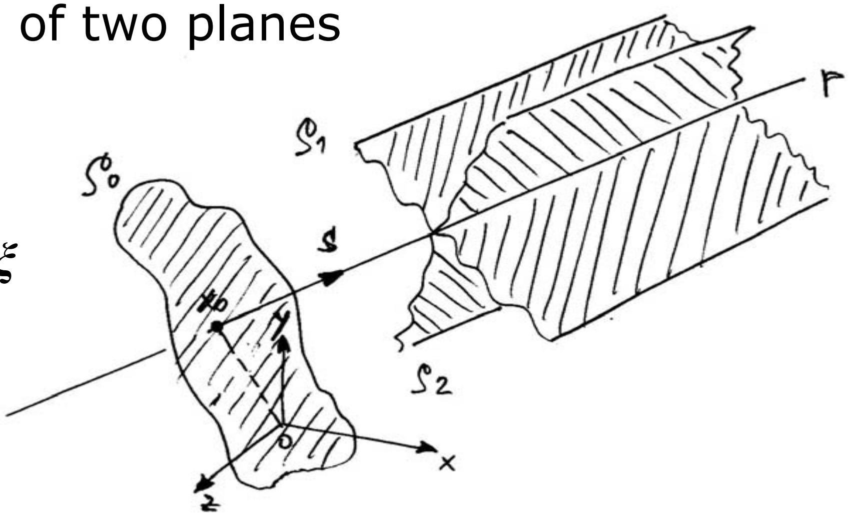
Simple solution:

- Translate planes $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ so the ξ is in the origin
- Compute intersection of two planes i.s. $\boldsymbol{x}_0$ and $\boldsymbol{s}$
- Translate $\boldsymbol{x}_0$ using $\boldsymbol{T}^{-1}$

Again – an elegant solution, simple formula supporting matrix-vector architectures like GPU and parallel processing

Solution DETAILS next

ξ



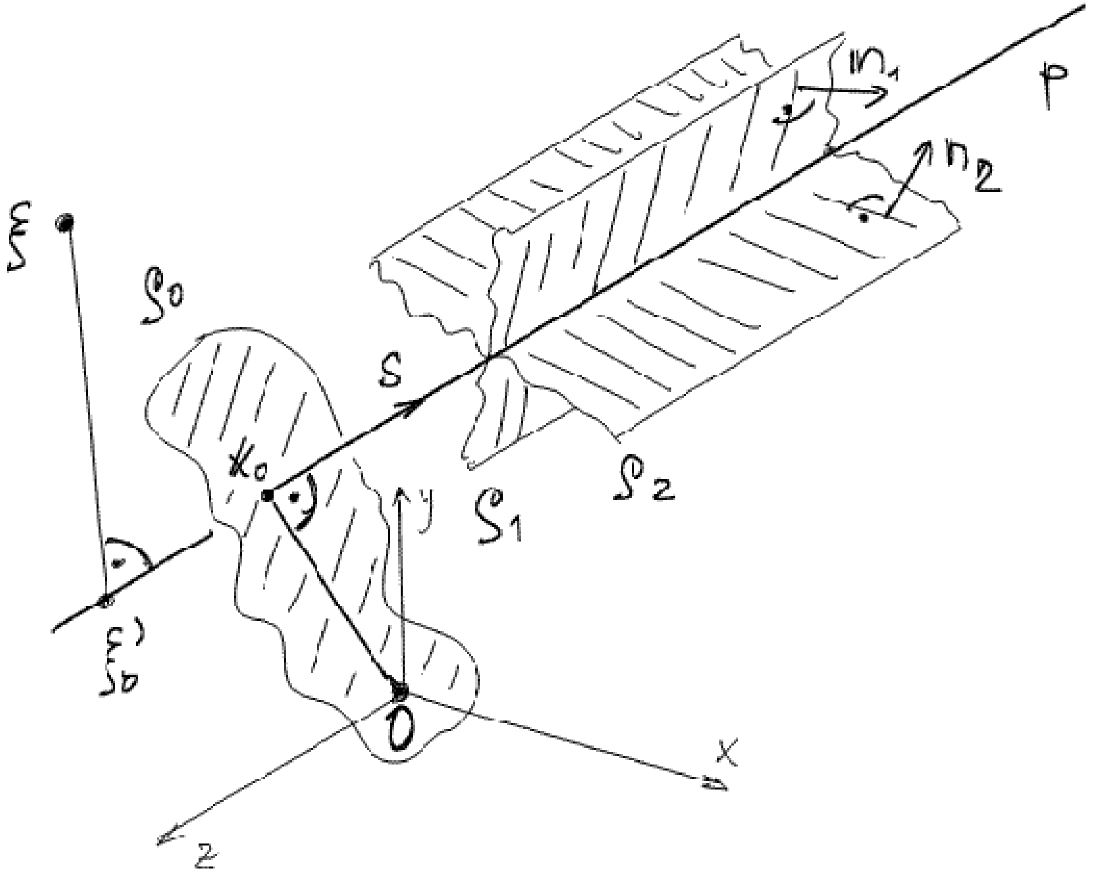
Known solution using Lagrange multipliers

$$\begin{bmatrix} 2 & 0 & 0 & n_{1x} & n_{2x} \\ 0 & 2 & 0 & n_{1y} & n_{2y} \\ 0 & 0 & 2 & n_{1z} & n_{2z} \\ n_{1x} & n_{1y} & n_{1z} & 0 & 0 \\ n_{2x} & n_{2y} & n_{2z} & 0 & 0 \end{bmatrix} \begin{pmatrix} p_x \\ p_y \\ p_z \\ \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} 2p_{0x} \\ 2p_{0y} \\ 2p_{0z} \\ \vec{p}_1 \cdot \vec{n}_1 \\ \vec{p}_2 \cdot \vec{n}_2 \end{pmatrix}$$

Krumm,J.: Intersection of Two Planes, Microsoft Research

The closest point to an intersection of two planes

In some applications we need to find a closest point on a line given as an intersection of two planes. We want to find a point $\boldsymbol{\xi}_0{}'$, the closest point to the given point $\boldsymbol{\xi}$, which lies on an intersection of two planes



$\boldsymbol{\rho}_1 \triangleq [\boldsymbol{n}_1^T : d_1]^T$ and $\boldsymbol{\rho}_2 \triangleq [\boldsymbol{n}_2^T : d_2]^T$

This problem was recently solved by using Lagrange multipliers and an optimization approach leading to a solution of a system of linear equations with 5 equations.

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Solution in the projective space

1. Translate the given point $\boldsymbol{\xi} = [\xi_x, \xi_y, \xi_z : 1]^T$ to the origin – matrix $\boldsymbol{Q}$
2. Compute parameters of the given planes $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ after the transformation as $\boldsymbol{\rho}'_1 = \boldsymbol{Q}^{-T}\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}'_2 = \boldsymbol{Q}^{-T}\boldsymbol{\rho}_2$,
3. Compute the intersection of those two planes $\boldsymbol{\rho}'_1$ and $\boldsymbol{\rho}'_2$
4. Transform the point $\boldsymbol{\xi}_0$ to the original coordinate system using transformation

$$\boldsymbol{n}_0 = \boldsymbol{n}_1 \times \boldsymbol{n}_2 \qquad \boldsymbol{\rho}_0 \triangleq [\boldsymbol{n}_0^T : 0]^T \qquad \boldsymbol{\xi}_0 = \boldsymbol{\rho}_1 \times \boldsymbol{\rho}_2 \times \boldsymbol{\rho}_0 \qquad \boldsymbol{\xi}'_0 = \boldsymbol{Q}^{-1}\,\boldsymbol{\xi}_0$$

$$\boldsymbol{Q} = \begin{bmatrix} 1 & 0 & 0 & -\xi_x \\ 0 & 1 & 0 & -\xi_y \\ 0 & 0 & 1 & -\xi_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \boldsymbol{Q}^{-T} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \xi_x & \xi_y & \xi_z & 1 \end{bmatrix} \qquad \boldsymbol{Q}^{-1} = \begin{bmatrix} 1 & 0 & 0 & \xi_x \\ 0 & 1 & 0 & \xi_y \\ 0 & 0 & 1 & \xi_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

It is simple, easy to implement on GPU.

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Curves and surfaces

Rational Bézier curve – Euclidean $\boldsymbol{X} = [X, Y, Z]^T$

$\boldsymbol{X}(t) = \frac{\sum_{i=0}^{n} B_i^n(t) w_i q_i}{\sum_{i=0}^{n} B_i^n(t) w_i}$ $\qquad 0 \leq t \leq 1 \qquad$ $B_i^n \binom{n}{i} (1-t)^{n-1} t^i$

1st derivative $\qquad \left[\frac{a}{b}\right]' = \frac{a'b - ab'}{b^2} \qquad$ quite complicated

Projective $\boldsymbol{x} = [x, y, z: w]^T$

$\boldsymbol{x}(t) = \sum_{i=0}^{n} B_i^n(t) q_i \qquad \boldsymbol{x}'(t) = \sum_{i=0}^{n} B_i^n(t) q_i$

How simple !

Computation in Projective Space

Disadvantages

- Careful handling with formula as the projective space
- "Oriented" projective space is to be used, i.e. $w \geq 0$;
  HW could support it
- Exponents of homogeneous vectors can overflow
  - exponents should be normalized; HW could support it
    unfortunately not supported by the current hardware
  - P_Lib – library for computation in the projective space - uses SW solution for normalization on GPU (C# and C++)

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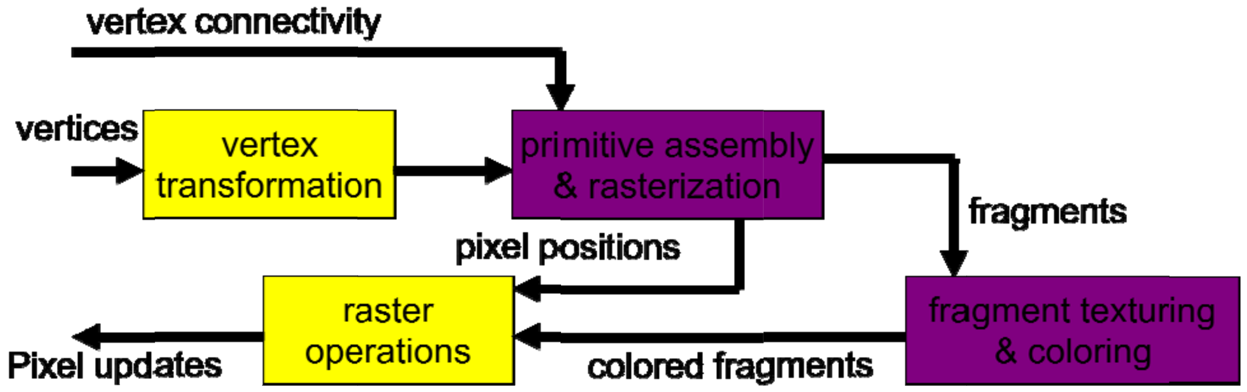
Computation in Projective Space

Advantages

- “Infinity” is well represented
- No division operation is needed, a division operation can be hidden to the homogeneous coordinate
- Many mathematical formula are simpler and elegant
- One code sequence solve primary and dual problems
- Supports matrix−vector operations in hardware – like GPU etc.
- Numerical computation can be faster
- More robust and stable solutions can be achieved
- System of linear equations can be solved directly without division operation, if exponent normalization is provided

Implementation aspects and GPU

- GPU (Graphical Processing Unit) -optimized for matrix-vector, vector-vector operation – especially for $[x,y,z:w]^T$
- Native arithmetic operations with homogeneous coordinates – without exponent "normalization"
- Programmable HW – parallel processing



Implementation aspects and GPU

4D cross product can be implemented in Cg/HLSL on GPU (not optimal implementation) as:

```
float4 cross_4D(float4 x1, float4 x2, float4 x3)
{ float4 a; # simple formula #
   a.x=dot(x1.yzw, cross(x2.yzw, x3.yzw));
   a.y=-dot(x1.xzw, cross(x2.xzw, x3.xzw));
   a.z=dot(x1.xyw, cross(x2.xyw, x3.xyw));
   a.w=-dot(x1.xyz, cross(x2.xyz, x3.xyz));
   return a;
}
# more compact formula available #
```

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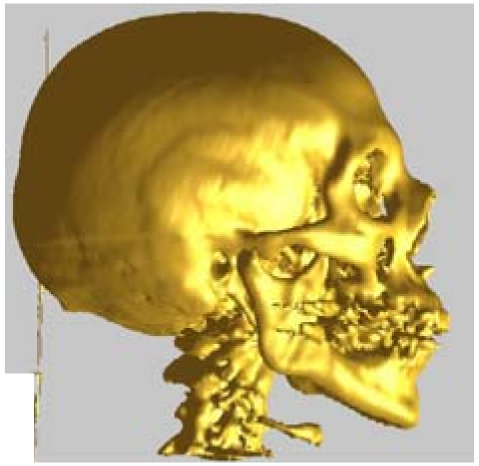
Appendix

Data processing - main field in computer science

Data processing itself can be split to two main areas:

- **processing of textual data**
  limited interval of values, unlimited dimensionality
  [a char as one dimension -
  Methionylthreonylthreonylglutaminylarginyl..isoleucine 189,819 chars]
  No interpolation is defined
- **processing of numerical data**
  unlimited interval of values,
  limited dimensionality – usually 2 or 3
  Interpolation **can be used**



	Textual	Graphical
Dim	∞	2, 3
Interval	0-255(ASCII)	(-∞, ∞)

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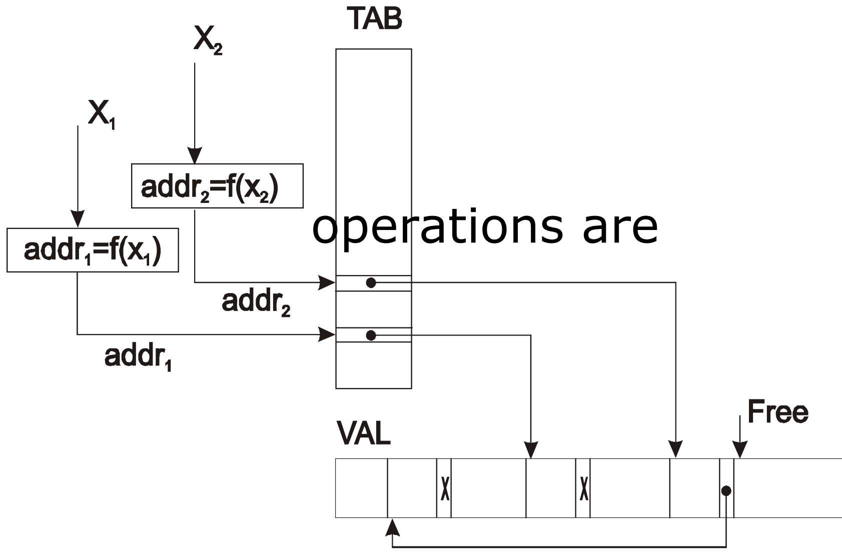
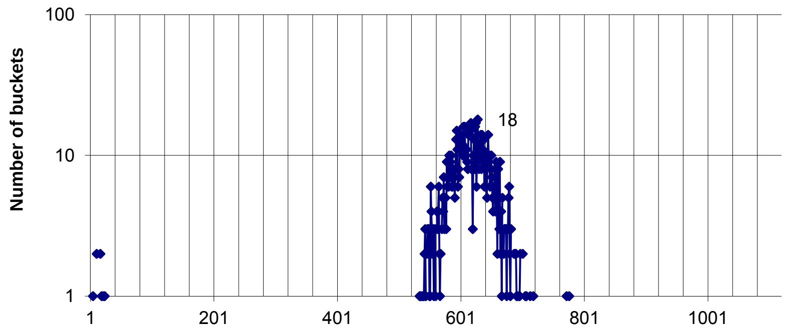
Hash functions

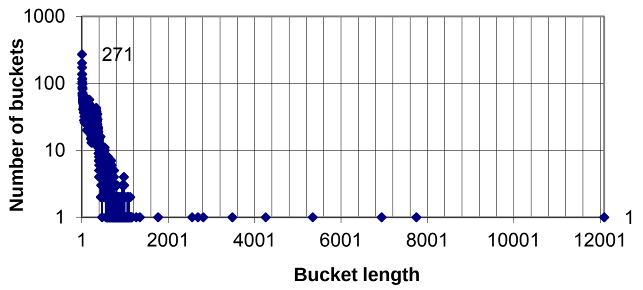
- usually used for textual data processing
- prime numbers and modulo operations are used

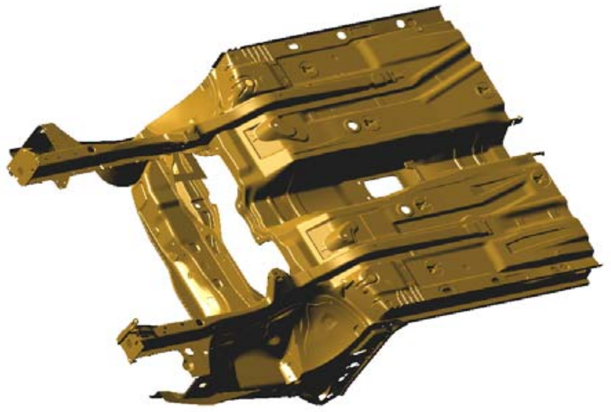
Usual form $Addr = \lfloor 3x + 5y + 7z \rfloor \mathbf{mod}\, size$

multiplication int * float needed









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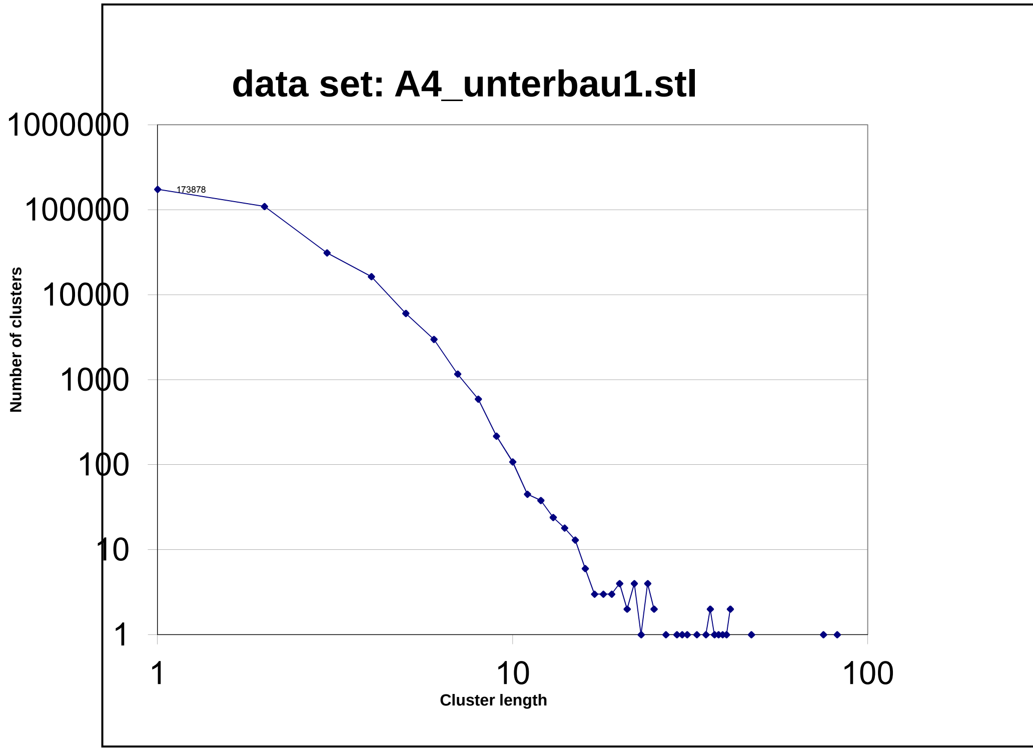
If the hash function is constructed as

$$Addr = \lfloor C(\alpha x + \beta y + \gamma z) \rfloor \mathbf{mod}\ m$$

where α, β, γ are “irrational” numbers and $m = 2^k$ better distribution is obtained
=> much faster processing.



C is a “magic” constant which maps the expression for $< x_{min}, x_{max} >$ to $(0, maxint)$

$$Addr = \lfloor C(\alpha x + \beta y + \gamma z) \rfloor \mathbf{and}\ (2^{\mathrm{k}} - 1)$$

- Hradek,J., Skala,V.: Hash Function and Triangular Mesh Reconstruction, Vol.29, No.6., pp.741-751, Computers&Geosciences, Pergamon Press, ISSN 0098-3004, 2003

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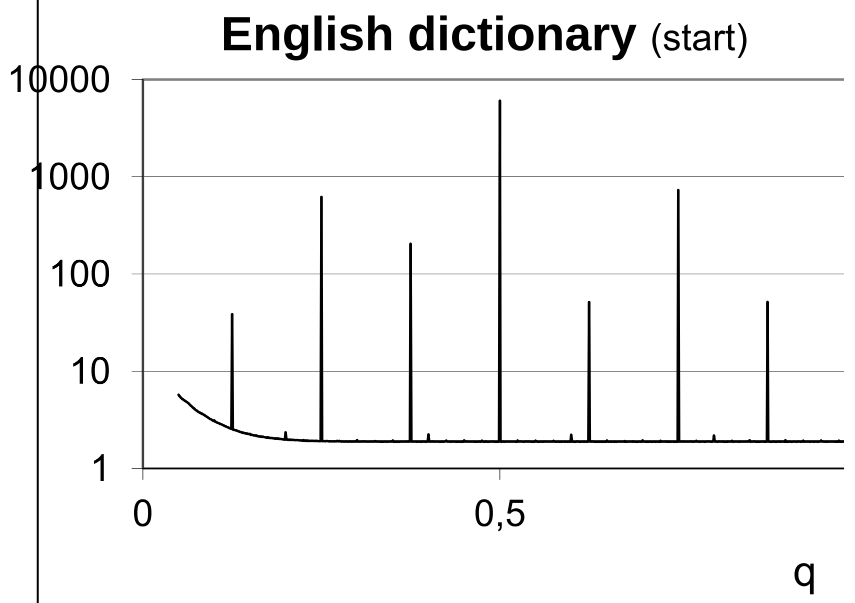
Textual processing



The has function is constructed as

$$h(\boldsymbol{x}) = \left(C * \sum_{i=1}^{L} q^i x_i \right) \boldsymbol{mod}\, m$$

q „irrational" $0 < q < 1$

$m = 2^k - 1$

Both geometrical and textual
hash function design have the same approach coefficients are "irrational" and no division operation is needed.

Some differences for Czech, Hebrew, English, German, Arabic, … languages and "chemical" words.

Summary and conclusion

We have got within this course an understanding of:

- projective representation use for geometric transformations with points, lines and planes
- principle of duality and typical examples of dual problems, influence to computational complexity
- intersection computation of two planes in E3, dual Plücker coordinates and simple projective solution
- geometric problems solution with additional constrains
- intersection computations and interpolation algorithms directly in the projective space
- barycentric coordinates computation on GPU
- avoiding or postponing division operations in computations

Projective space representation supports matrix-vector architectures like GPU – faster, robust and easy to implement algorithms achieved

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?? Questions ??

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