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Synthesis of Reeb Graph and Morse Operators from Level Sets of a Boundary Representation

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Abstract

In the context of Industrie 4.0, it is necessary for several applications, to encode characteristics of a Boundary Representation of a manifold $\mathcal{M}$ in an economical manner. Two related characterizations of closed B-Reps (and the solid they represent) are (1) medial axis and (2) Reeb Graph. The medial axis of a solid region is a non-manifold mixture of 1-simplices and 2-simplices and it is expensive to extract. Because of this reason, this manuscript concentrates in the work-flow necessary to extract the Reeb Graph of the B-Rep. The extraction relies on (a) tests of geometric similarities among slices of $\mathcal{M}$ and (b) characterization of the topological transitions in the slice sequence of $\mathcal{M}$. The process roughly includes: (1) tilt of the B-Rep to obtain an unambiguous representation of the level sets of $\mathcal{M}$,(2) identification and classification of the topological transitions that arise between consecutive level sets, (3) sample of Reeb graph vertices inside the material regions defined by the level sets, (4) creation of Reeb graph edges based on the type of topological transition and the 2D similarity among material regions of consecutive levels. Although the Reeb Graph is a topological construct, geometrical processing is central in its synthesis and compliance with the Nyquist-Shannon sampling interval is crucial for its construction. Future work is needed on the extension of our methodology to account for manifolds with internal voids or nested solids.

CCS Concepts

• ***Computing methodologies*** → ***Computer graphics; Shape analysis; Volumetric models;***

1. Introduction

The encoding of geometry and topology characteristics of a Boundary Representation (B-Rep) in a computationally economical manner is a useful process in several fields such as medical imaging, computer graphics and computational mechanics [GSBW11]. Two of the most commonly used characterizations of a closed manifold are (1) medial axis and (2) Reeb Graph. The Reeb Graph in particular is used in the analysis of large data sets, such as: the efficient classification and segmentation of large point clouds [WYLL21], mesh segmentation oriented towards topological optimization [MG21], CAD model segmentation [HR20, SJ17], shape similarity and matching [MBH12] and data abstraction from large data sets [NBPF11]. For a detailed description of applications of Reeb Graphs in computer graphics see [BGSF08].

The goal of this manuscript is to introduce the necessary steps to synthetize the Reeb Graph from the Boundary Representation of a closed manifold $\mathcal{M}$. The proposed methodology relies on (a) the identification of the critical points of the slice-driven Morse function defined on $\mathcal{M}$ and (b) the synthesis of connectivity between critical points based on tests of geometric similarities between slices of $\mathcal{M}$ and the type of topological transitions.

The *Medial Axis* of a compact 3D region $\Omega \subset \mathbb{R}^3$ is defined as the set of all points $p \in \Omega$ such that the closest point in the boundary $\partial\Omega$ is not unique. The medial axis is a non-manifold mixture of 1-simplices and 2-simplices. The extraction of the medial axis of a closed manifold is computationally expensive and therefore unsuitable for applications that require real time interaction.

The *Reeb Graph* [Ree46] is a way to encode the topological characteristics of a closed manifold in an efficient manner. The Reeb graph depends on the characteristics of the level sets determined by a slicing-driven function on a closed manifold $\mathcal{M}$. Slicing a closed 2- or 3-manifold mesh is to compute level sets of a height function $f : \mathcal{M} \to \mathbb{R}$ with $f(x,y,z) = z$. The preimage f^{-1} of such function at a point c is known as a level set of f.

The topological characteristics of manifold $\mathcal{M}$ are determined by the critical points of the function f defined on $\mathcal{M}$. To avoid ambiguity in the level sets, the height function f defined on $\mathcal{M}$ must be a *Morse* function. A *Morse* function is characterized by its critical points. A critical point of f is a point $p \in \mathcal{M}$ such that its tangent gradient $\nabla_{\mathcal{M}} f(p)$ is zero. A critical point is degenerate if its tangent Hessian matrix $\mathbf{H}_{\mathcal{M}} f(p)$ is degenerate, that is, if its matrix determinant is zero.

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Morse Function: Let $\mathcal{M}$ be a closed and oriented 2- or 3-manifold without inner cavities embedded in $\mathbb{R}^3$, and consider a twice differentiable function $f\colon \mathcal{M} \to \mathbb{R}$. The function f is a *Morse function* if all critical points of f are non-degenerate:

$$\forall p \in \mathcal{M} : \nabla_{\mathcal{M}} f(p) = 0 \to \det\left(\mathbf{H}_{\mathcal{M}} f(p)\right) \neq 0 \tag{1}$$

Given a *Morse* function f defined on manifold $\mathcal{M}$, it is then possible to define the Reeb Graph of f, denoted as $R(f)$.

Reeb Graph: Let $f : \mathcal{M} \to \mathbb{R}$ be continuous and call a component of a level set a contour. Two points $p, q \in \mathcal{M}$ are equivalent if they belong to the same connected component of $f^{-1}(c)$ with $c = f(p) = f(q)$. The Reeb Graph of f, $R(f) = \mathbb{X}_{\sim}$, is the quotient space defined by this equivalence relation.

There is a continuous map $\psi : \mathcal{M} \to R(f)$. Point $u \in R(f)$ is a *node* if $\psi^{-1}(u)$ contains a critical point, that is, if u is the image of a critical point of f under ψ.

The rest of the manuscript is structured as follows: Section 2 presents the literature review regarding the Reeb Graph extraction on manifolds, Section 3 presents our proposed methodology, Section 4 shows the application of the proposed methodology to example data sets and Section 5 concludes the manuscript.

2. Literature Review

Two of the most commonly used characterizations of a closed manifold are (1) medial axis and (2) Reeb Graph. The calculation of the medial axis of a 3D region is a computationally expensive problem [dMPF09, RG03], making it unsuitable for applications that require real time interaction. The Reeb Graph was introduced to graphics applications by Shinagawa et al. [SK91].

Available methods for the extraction of Reeb Graph can be classified according to the choice of the Morse function $f : \mathcal{M} \to \mathbb{R}$ that encodes the topological information of the manifold [Bia04]. Some functions used include the height function [MPRSC*20, BGSF08], the geodesic distance from a seed vertex [HZP14] and distance from center of mass [BMM*03]. The height function approach imposes the lowest computational cost of all three options but requires an adequate definition of the slicing that defines the function f. The main advantage of such a function is the independence from translations and uniform scaling. However, the height function is not independent from rotations. The approaches based on distance from barycenter and geodesic distance from a seed vertex impose greater computational cost than the height function method [BMM*03], but with the advantage of independence with respect to rotations.

Given a slicing-driven height function, available methods differ on how to find the connectivity of the Reeb Graph. Standard methods rely solely on proximity between level sets and handle classification, such as the ones in [SK91, HWW10]. Other authors have proposed to use heat-based mesh segmentation [HZP14] or triangular mesh collapse [HWW10] to link together the nodes of the Reeb Graph. Sweep algorithms are also used to find the connectivity of the Reeb Graph, such as the one in [CMEH*04]. Some authors have explored the 2D shape similarity analysis between the polygonal regions denoted by a connected component of a level set as a filter to establish connectivity between slices [RC01, RCG*05] in the surface reconstruction context. The proximity-only solutions are unreliable to produce correct results in complex topological transitions [SK91]. The addition of a shape similarity filter increases the reliability of the connectivity extraction by ensuring the correctness of each ancestor-descendant relationship between level sets. Other approaches, such as the one in [SSJ11], are heavily dependant on the mesh representation of the manifold, entailing problems regarding the mesh density and the computational cost of mesh segmentation. Other works have focused on the definition and extraction of discrete Reeb Graphs on voxelized domains [BB18] or the reduction of the topological complexity of the extracted graphs [DSMP16].

The Reeb Graph is able to adequately reflect the topological structure only of manifolds with no inner voids. When inner voids are present, the Reeb Graph fails to univocally capture the topological structure of the manifold [SJ15, EHP08]. The reason for this is that the Reeb Graph is sensible only to topological changes that affect the number of connected components in the level set (i.e. in the cross section). The introduction of inner contours does not change the number of connected components in a level set.

2.1. Conclusion of Literature Review

Reeb Graph extraction methods can be classified according to the nature of the function $f : \mathcal{M} \to \mathbb{R}$ that encodes the topological characteristics of the manifold. The most commonly used function is the height function. It allows for easy implementation and low computational cost at the setback of being dependant on the orientation of the manifold in 3D space.

The synthesis of the edges of the Reeb Graph is also approached using different methods. Proximity-only solutions are unreliable for automatic extraction of the edges and other approaches entail high computational costs. Some authors have explored 2D shape similarity as an additional filter to improve reliability of level set connectivity in other contexts such as surface reconstruction. The Reeb Graph is limited to 2-manifolds or 3-manifolds without inner cavities, since it is only sensible to topological changes that affect the number of connected components.

To encode 3D shape, the medial axis computation is extremely expensive if directly addressed. On the other hand, the Reeb Graph by itself presents the aforementioned limitations. Because of this reason, this manuscript presents the first steps in supplementing the Reeb Graph with geometrical information, thus allowing in the future a reasonable encoding of 3D shape with inner voids characteristics.

3. Methodology

We propose a methodology to extract the Reeb Graph of a given Boundary Representation of a 2- or 3-manifold without inner cavities $\mathcal{M}$ embedded in $\mathbb{R}^3$. Our algorithm can be summarized in the following steps:

1. *Level set extraction*: Tilt of the B-Rep to obtain an unambiguous representation of the level sets of $\mathcal{M}$.
2. *Nodes definition*: Sampling of the material regions denoted by the obtained level sets of $\mathcal{M}$ to obtain the nodes of the Reeb Graph.

3. *Edges definition*: Synthesis of the connectivity (edges) of the Reeb Graph according to the criteria of shape similarity and the type of topological transition.

3.1. Level sets extraction

Given a Boundary Representation of a 2- or 3-manifold without inner cavities $\mathcal{M}$ embedded in $\mathbb{R}^3$ (Fig. 1), a height function $f : \mathcal{M} \to \mathbb{R}$ is defined on the manifold driven by a planar slicing with a set of planar surfaces Π parallel to the $x-y$ plane. The orientation of the manifold $\mathcal{M}$ in $\mathbb{R}^3$ must be one in which the function induced by the planar slicing is a *Morse* function (see Eq. 1). The fact that the function f defined on $\mathcal{M}$ is Morse ensures the unambiguity of the level sets of $\mathcal{M}$ retrieved from such a mapping.

A level set $\Pi_c : f^{-1}(c)$ defined by the preimage of the Morse function f can contain one or more contours (connected components). For example, in Fig. 1, the level set Π_{c_i} has one contour and the level set Π_{c_j} has two contours. The contour population between level sets evolve as a result of changes in the cross-section composition of $\mathcal{M}$.

The distance between the slices is subjected to compliance with the Nyquist-Shannon principle in all directions. The technician must decide which level of geometric detail d is to be captured. The sampling distance should be less than $d/2$ (in all directions). Therefore, there is no universal sampling rate. For example, if the designer wants to preserve very close cavities as separate ones, the sampling distance (in all directions) must be set up as half of the minimal separation among holes, or less

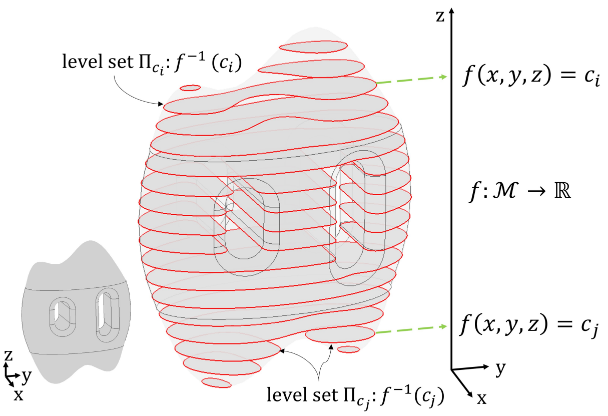

Figure 1: *Definition of Morse function $f : \mathcal{M} \to R$ on $\mathcal{M}$. Level sets are obtained by the preimage of f.*

3.2. Nodes definition

A set $\mathcal{P}$ of material points is obtained by sampling each polygonal region denoted by the contours of each level set. Each point $p \in \mathcal{P}$ represents a polygonal region (connected component) inscribed within one or more contours. As stated before (see Section 1), the vertices V of the Reeb graph $R(f) = (V,E)$ are the points in $\mathcal{P}$ such that they are critical points (i.e. they belong to a critical level set).

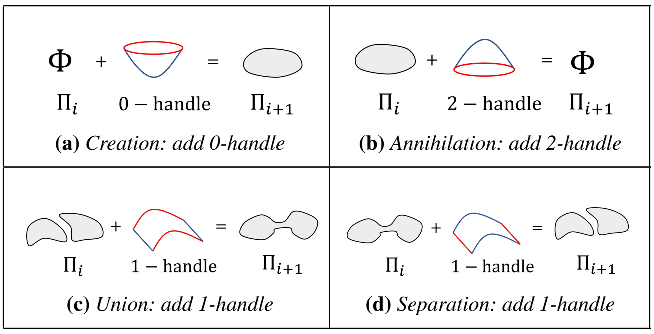

Figure 2: *Effect of Morse operators (handles) on the contour population between level sets Π_i and Π_{i+1}.*

3.3. Edges definition

Before establishing a connectivity between the nodes of the Reeb Graph, the topological changes in the level set sequence must be identified and classified. A set of Morse operators known as *handles* govern the evolution of the contour population. Each *handle* represents a topological change in the manifold $\mathcal{M}$ and a change in the number of contours (connected components) between level sets. The application of handle operators can be classified as follows (Fig. 2): (a) a 0-handle creates a new contour from the empty set, (b) a 2-handle annihilates a contour and (c) a 1-handle either separates a contour into two different contours or unites two contours into a single contour.

The occurrence of handles in the level set sequence dictates whether a level set is critical or not. A level set is critical if there is a change in the contour population with respect to the previous or the next level set in the sequence. Fig. 3 shows the critical level sets in the sequence for manifold $\mathcal{M}$ and the type of handle operator that acts upon the level set sequence in each step.

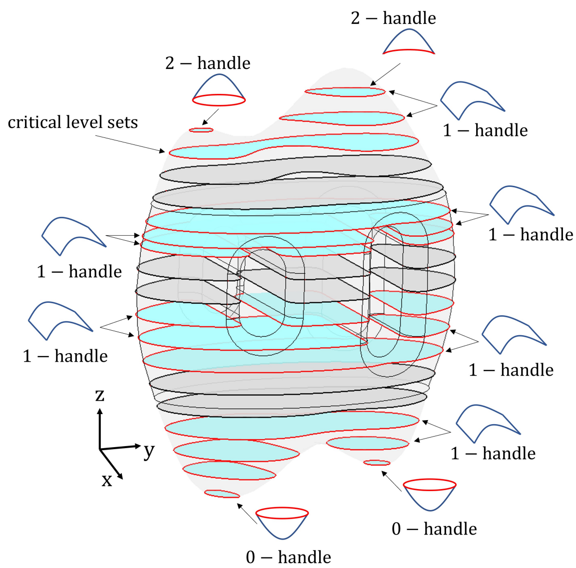

Figure 3: *Critical level sets on manifold $\mathcal{M}$ with handles.*

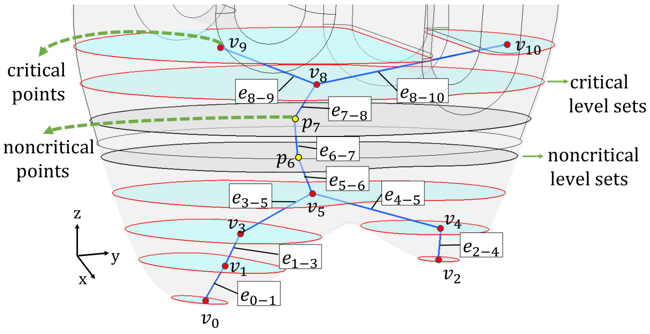

Figure 4: *Connectivity between level sets to synthetize Reeb graph* $R(f) = (V,E)$.

Even though the slicing provides the vertices of the Reeb graph, the connectivity of the Reeb graph does not unequivocally follows from the slicing. We propose an heuristic to find the edges E of Reeb graph $R(f) = (V,E)$ by following the level set sequence and connecting two vertices in neighboring level sets according to (a) the handle operator that acts on the contour population between the neighboring level sets and (b) shape similarity between polygonal regions. In this heuristic, the non-critical level sets are necessary to synthetize the connectivity of the Reeb graph.

For example, in Fig. 4, to obtain the connectivity between vertices v_5 and v_8 it is necessary to take into account noncritical points p_6 and p_7 (edges $e_{5-6}, e_{6-7}, e_{7-8}$). Each edge is labeled with a handle operation according to the contour population evolution. Edges e_{3-5} and e_{4-5} are 1-handle edges since they connect a level set with two contours (represented by vertices v_3 and v_4) with a level set with only one contour (represented by vertex v_5).

Once all vertices and edges are obtained, the Reeb graph $R(f)$ for a Morse function $f : \mathcal{M} \to \mathbb{R}$ encodes the characteristics of the Boundary Representation of the manifold $\mathcal{M}$, as seen in Fig. 5.

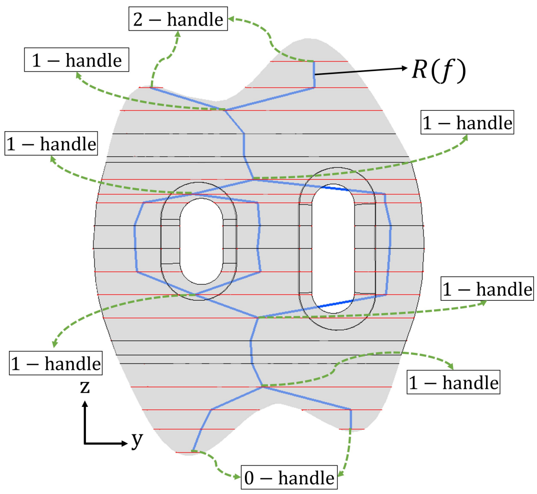

Figure 5: *Reeb graph* $R(f)$ *for the example manifold* $\mathcal{M}$.

4. Results

Figure 6 shows the Reeb Graph synthetized for two example data sets. Figures 6a and 6b shows the Reeb Graph for the *hands* dataset. Figures 6c and 6d shows the Reeb Graph for the *elephant* dataset. In both examples the synthetized Reeb Graph correctly captures the topological transitions that occur through the Boundary Representation. Notice that, since our methodology is geometrically-driven, the Reeb graph resembles the geometry of the manifold $\mathcal{M}$.

Even though there is not a preferred way of drawing the Reeb Graph (connections between nodes could take any shape), the fact that the connections resemble the geometry of the manifold $\mathcal{M}$ is useful towards the use of the Reeb Graph for the computation of the medial axis of the manifold $\mathcal{M}$.

5. Conclusions

This manuscript presents a workflow for the synthesis of a Reeb Graph encoding for a solid region in $\mathbb{R}^3$ denoted by its Boundary Representation $\mathcal{M}$. The Reeb Graph for $\mathcal{M}$ is a well known topological entity. However, its geometrical realization presents challenges and variations. Our approach starts with the rotation of $\mathcal{M}$ to obtain a Morse function $f : \mathcal{M} \to \mathbb{R}$ on the manifold, with $f(x,y,z) = z$ for point (x,y,z) in $\mathcal{M}$. Morse-compliance guarantees that level sets of f unambiguously determine the material regions of $\mathcal{M}$ on each slice. The nodes of the Reeb Graph (non-degenerate critical points of f) are detected by registering the topological changes (i.e. classifying the Morse handles) in the level sets of a Nyquist-Shannon equispaced slicing of $\mathcal{M}$. The Reeb Graph admits several edges for each pair of nodes. Detection and per-slice-tracing of these edges is achieved by using the handle classification and 2D shape similarity among level sets.

This geometry-driven methodology correctly synthesizes the Reeb Graph of the example B-Reps. It results in Reeb Graph representations faithful to the geometrical characteristics of $\mathcal{M}$, and not only to its topological features. Future work is needed in: (a) the extension of the methodology to obtain a topological representation of manifolds in $\mathbb{R}^3$ with inner cavities and (b) trying to achieve independence of the slicing with respect to the orientation of the manifold in $\mathbb{R}^3$.

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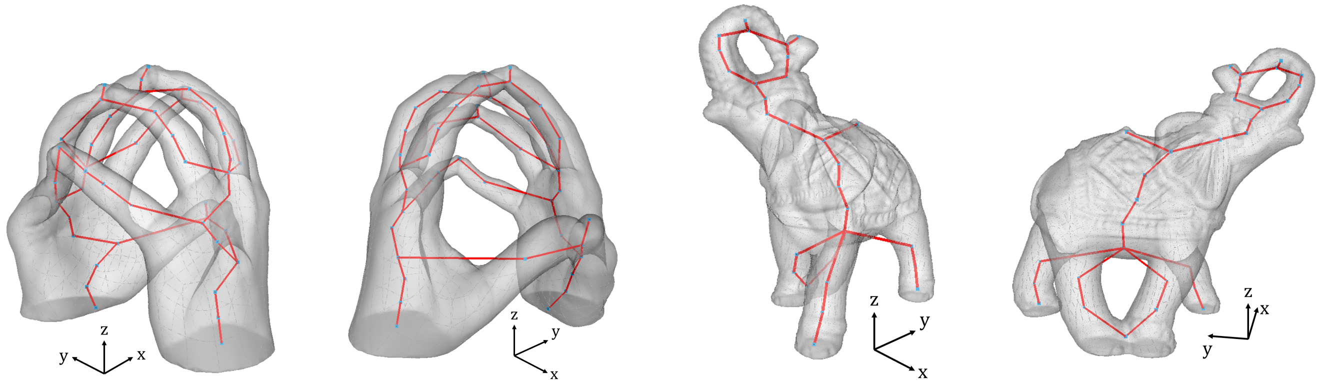

(a) *Reeb Graph on hands dataset. Viewpoint 1.*
(b) *Reeb Graph on hands dataset. Viewpoint 2.*
(c) *Reeb Graph on elephant dataset. Viewpoint 1.*
(d) *Reeb Graph on elephant dataset. Viewpoint 2.*

Figure 6: *Synthesis of the Reeb Graph representation on example manifolds.*

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