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# Triangulations with Locally Optimal Steiner Points

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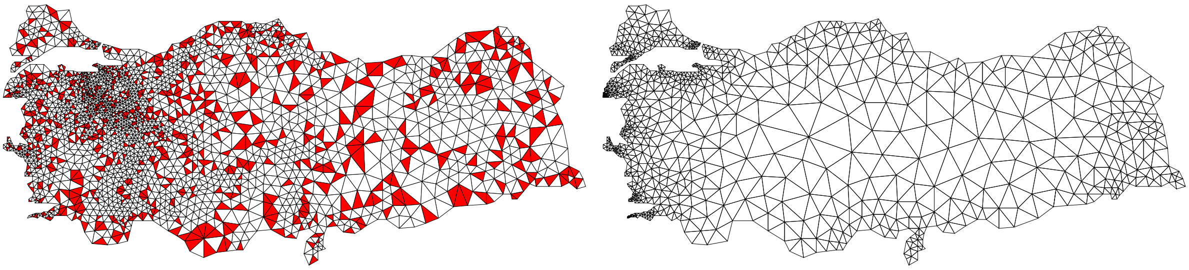

**Figure 1:** *The original Delaunay refinement algorithm does not terminate for constraint angles larger than* 34°. *Our algorithm computes quality triangulations for constraint angles up to* 42°. *For example, when the constraint angle is* 40°, *the original Delaunay refinement chokes with bad (shaded/red) triangles (left), where as our algorithm computes a nicely graded triangulation where all angles are larger than or equal to the constraint angle (right).*

**Abstract**

*We present two new Delaunay refinement algorithms, the second an extension of the first. For a given input domain (a set of points in the plane or a planar straight line graph), and a threshold angle $\alpha$, the Delaunay refinement algorithms compute triangulations that have all angles at least $\alpha$. Our algorithms have the same theoretical guarantees as the previous Delaunay refinement algorithms. The original Delaunay refinement algorithm of Ruppert is proven to terminate with size-optimal quality triangulations for $\alpha \leq 20.7^\circ$. In practice, it generally works for $\alpha \leq 34^\circ$ and fails to terminate for larger constraint angles. The new Delaunay refinement algorithm generally terminates for constraint angles up to $42^\circ$. Experiments also indicate that our algorithm computes significantly (almost by a factor of two) smaller triangulations than the output of the previous Delaunay refinement algorithms.*

Categories and Subject Descriptors (according to ACM CCS): F.2.2 [Nonnumerical Algorithms and Problems]: Geometrical problems and computations I.3.5 [Computational Geometry and Object Modeling]: Geometric algorithms, languages, and systems

## 1. INTRODUCTION

We revisit the following well-known two-dimensional geometric optimization problem: *Compute the smallest size triangulation of a given two dimensional domain (collection of points and/or segments) such that all the triangles in the triangulation are of good quality.* Naturally, we are allowed to introduce points (called the *Steiner points*) additional to the input points. This problem is also known as the simplicial mesh generation problem, or the quality Steiner triangulation problem [Ede01, She02, TW00]. Quality constraint of the problem is motivated by the numerical methods used in engineering applications where these triangulations are heavily used. A triangle is said to be *good* if its smallest angle is bounded from below.Small size objective is crucial for the applications as well, for obvious efficiency reasons.

Several (approximation) algorithms have been suggested for this problem. Quadtree refinement algorithms [BEG94] and Delaunay refinement algorithms [Che89, Rup93, Üng04]

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provide similar theoretical guarantees. In general for a specific algorithm, these guarantees can be stated as: *Given a two dimensional domain and a threshold angle* $\alpha \leq \gamma$, *the algorithm computes a triangulation of the input domain whose size is within a constant factor of the optimal such that all the angles of the triangulation are at least* $\alpha$. The angle $\gamma$ should be thought of as the theoretical limit of an algorithm. The exact value of the $\gamma$ depends on the approach/algorithm as well as the input type. For instance, the Delaunay refinement algorithm of Ruppert [Rup93] is proven to compute good triangulations for $\alpha < \gamma = 30^\circ$ on point sets and for $\alpha < \gamma = 20.7^\circ$ on planar straight line graphs.

In practice, the lack of theoretical guarantee for a mesh refinement algorithm, when $\alpha > \gamma$, is observed as a never-ending refinement process. Steiner points are iteratively inserted until the computer reaches out of its numerical precision capacity or its memory. Shewchuk's experimental study revealed that in practice Delaunay refinement algorithm works better than its theoretical guarantee [She96]:

> "Ruppert [Rup93] proves that this procedure halts for angle constraint of up to $20.7^\circ$. In practice, the algorithm generally halts with an angle constraint of $33.8^\circ$, but often fails to terminate given an angle constraint of $33.9^\circ$. It would be interesting to discover why the cutoff falls there."

There has been attempts to explain this cutoff [MPW05]. Here, we take a different strategy and rather than explaining this limitation, we remove it. We present a new Delaunay refinement algorithm which sets a new cut-off. Our refinement algorithm terminates with good triangulations for constraint angles up to $42^\circ$.

### 1.1. Previous Work

#### 1.1.1. Delaunay refinement

Delaunay refinement method involves first computing an initial Delaunay triangulation of the input domain, and then iteratively adding points called *Steiner points* to improve the quality of the triangulation. Various types of Steiner points are studied in the literature, which we review below.

*Circumcenters.* Circumcenters of bad triangles, as studied by many [Che89, Rup93, She02], is a natural choice for improving the quality of a (Delaunay) triangulation through iterative refinement. Insertion of the circumcenter of a bad triangle, surely removes the bad triangle (thanks to the empty circle property of Delaunay triangulations.)

*Sinks.* Edelsbrunner and Guoy [EG01] suggested inserting *sinks* of bad triangles, which are circumcenters of acute triangles. For each bad triangle, an iterative walk in the triangulation each time crossing the edge opposite to the unique obtuse angle leads to its sink. Sinks are at the local maxima of the local feature size function. Note that the sink of a bad triangle can be quite far away from the triangle. Hence, a bad triangle may remain in the triangulation even after its sink is inserted.

*Offcenters.* Üngör [Üng04] introduced another type of Steiner points, called *off-centers*, as an alternative to circumcenters. Offcenters and circumcenters, differ from each other only for "very" bad triangles (those with smallest angle at most $\alpha/2$, and are the same for almost good triangles (those with the smallest angle between $\alpha/2$ and $\alpha$. In practice, off-center insertion algorithm results in significant reduction in the output size [Üng04]. Off-centers are also numerically more stable than circumcenters and facilitates more robust software. The idea of using offcenters also leads to the design of the first time-optimal Delaunay refinement algorithm [HPÜ05].

Recent years witnessed an inflation in the development and use of Delaunay refinement algorithms some of which are mentioned earlier [CGS06, EG01, HPÜ05, HMP06, Mil04, MPW05, She00, She02, Üng04]. Miller *et al.* proposed a time-efficient (but not time-optimal) Delaunay refinement algorithm [HMP06, Mil04]. Their algorithm currently lack experimental support to indicate its relevance in practice. Coll *et al.* [CGS06] studied Delaunay refinement within the context of mesh editing. However, currently the only published version of their work contains errors (as acknowledged by the authors in a recent personal communication) that make their experimental study inconclusive. These algorithms do not address the termination problem which is one of our primary focus in this study.

*Delaunay refinement software.* The popular mesh generation software is a robust implementation of the Delaunay refinement method [She96]. The software used Ruppert's circumcenter insertion algorithm [Rup93] in it first four releases and has been using Üngör's offcenter insertion algorithm [Üng04] in its latest two releases.

#### 1.1.2. Other mesh refinement algorithms

Quadtree methods provide similar guarantees as Delaunay refinement algorithms [BEG94]. However, in practice quadtree refined meshes are significantly larger than Delaunay meshes. Moreover, quadtree implementations lack the parameterization of the constraint angle.

There are other refinement strategies which are used in conjunction with Delaunay triangulations. For instance, the longest-edge propagation path (LEPP) methods use the midpoint of the longest edge of certain triangles as the Steiner point [RH99]. Theoretical bounds for these methods tend to be relatively weaker than that of the Delaunay refinement methods. Moreover, these methods also suffer from the termination problem in practice for constraint angles even smaller than $34^\circ$.

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### 1.2. Our idea and contribution.

All existing mesh refinement algorithms, including the circumcenter and the offcenter insertion variants of the Delaunay refinement, suffer from a so called *termination problem* for large values of $\alpha$. In this paper, we explore ideas to alleviate this problem.

We propose two simple algorithms, the second an extension of the first. The first algorithm suggests to locate Steiner points inside a particular neighborhood of bad triangles that are furthest away from existing vertices. One might wonder "Are such points not circumcenters (at least for the almost good triangles)?". It turns out the answer is "No!", for most of the bad triangles in a typical Delaunay triangulation. In Section 3, we classify this Steiner point description into four different types.

Our second algorithm incorporates a simple relocation scheme which becomes effective especially for large $\alpha$ values. Before attempting to insert a Steiner point for a bad triangle, we simply try fixing the bad triangle by relocating one of its vertices (if that vertex was inserted as a Steiner point).

These two simple ideas lead to the following as the main contributions of this paper:

- We give a definition of Steiner points which unifies the aforementioned Steiner point insertion strategies. The new rule automatically selects one of the four types of Steiner points, three of which (circumcenters, sinks, offcenters) are previously studied.
- With the new Steiner point definition, we extend the practical angle bound of the Delaunay refinement algorithms. While the previous Delaunay refinement algorithms fail to compute meshes, for constraint angles as small as $33.9^\circ$, the new algorithm computes good triangulations for constraint angles as high as $42^\circ$. On average, we extend the practical constraint angle bound by about $8^\circ$.
- The key idea in our Steiner point selection method is to stay away from the existing vertices. It turns out that circumcenters are not the best choice for this objective. Indeed, experiments give a surprising low percentage use for circumcenters among the four alternatives. We pick locations that result in longer edge lengths for the triangles of the mesh. As a result, the size of the output of our method is significantly smaller than that of the previous Delaunay refinement algorithms.
- We present a simple framework where a simple local smoothing (point relocation) strategy is integrated into the refinement algorithm.

## 2. MOTIVATION

We explore ways to improve the practical performance of Delaunay refinement method. Existing versions of Delaunay refinement method becomes impractical when meshes with large smallest angles are desired. Two reasons behind this are explained below.

### 2.1. Termination Problem

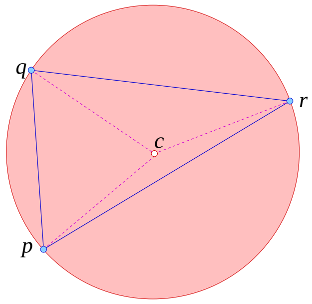


**Figure 2:** *When $\alpha > 30^\circ$, circumcenter insertion introduces shorter pairwise distances than existing ones, i.e.* $|pc| = |qc| = |rc| < |pq|$

Termination guarantee of the Delaunay refinement algorithms relies on a packing argument. This argument applies for small values of $\alpha$, (i.e. $\alpha \leq 30^\circ$) where circumcenter insertion does not gradually decrease the shortest pairwise vertex distance. For $\alpha > 30^\circ$, however, the iterative refinement process could introduce smaller and smaller features and may not terminate (See Figure 2.) In practice, such phenomenon starts occurring for $\alpha > 34^\circ$, regardless of the type and distribution of input data. See Figure 3 and Figure 4.

*Different Versions.* Any refinement algorithm that makes significant use of circumcenters is expected to suffer from the termination problem for threshold angles not much larger than $30^\circ$. Majority of the points inserted by most Delaunay refinement algorithms are circumcenters [Che89, EG01, Rup93, She02]. While the circumcenters are less frequently used by the offcenter insertion algorithm [Üng04], their percentage is significant enough that the practical performance bound remain roughly the same. Figure 3 illustrates how both `Triangle 1.4` which is an implementation of the circumcenter insertion, and `Triangle 1.6` which is an implementation of the offcenter insertion, suffer from the termination problem. It is important to observe the characteristic difference between these two methods as they get caught in the never-ending refinement process. This difference is due to the order the bad triangles are handled. The offcenter insertion algorithm works better when the bad triangles with the shortest edges are handled first. [HPÜ05]. On the other hand, the circumcenter insertion algorithm works better when the bad triangles with the smallest angles are handled first. When they terminate the offcenter insertion performs better than the circumcenter insertion, that is, it outputs meshes with fewer triangles. However, when interrupted in the never-ending refinement process, the evolving mesh of the offcenter refinement looks worse than that of the

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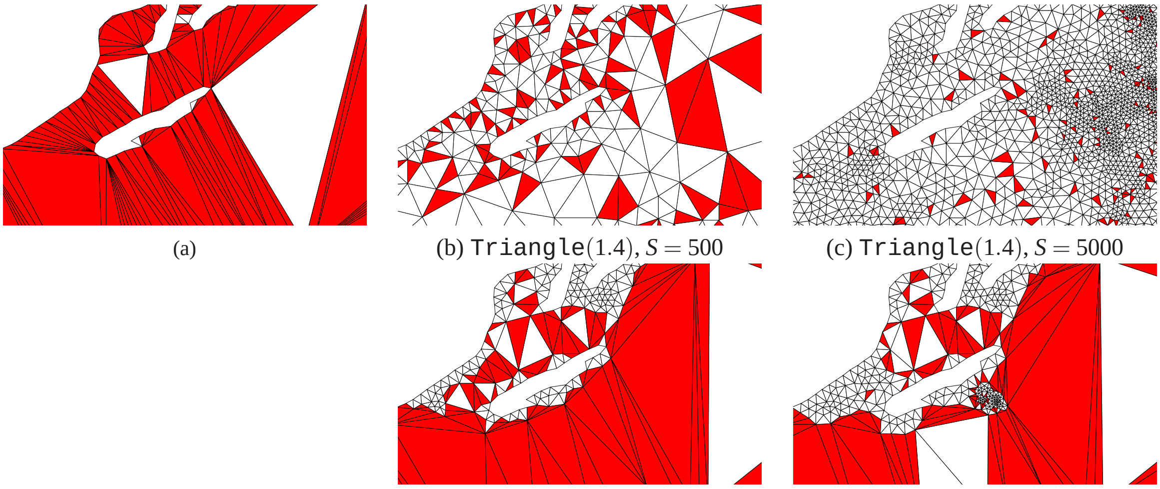
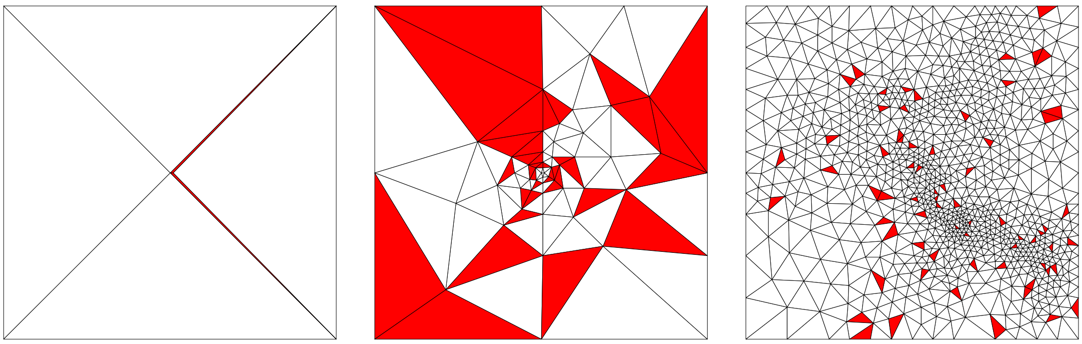


(a) (b) `Triangle`(1.4), $S = 500$ (c) `Triangle`(1.4), $S = 5000$

(d) `Triangle`(1.6), $S = 500$ (e) `Triangle`(1.6), $S = 750$

**Figure 3:** *Termination problem shown for the constraint angle $\alpha = 35.5^\circ$ on the Lake Superior data set (zoomed in). Bad triangles are shaded (red). (a) Delaunay triangulation of the input. (b) and (c) Evolving triangulation using the circumcenter insertion. (d) and (e) Evolving triangulation using the offcenter insertion. Neither of the two algorithms terminates. So, for this illustration, we interrupted their execution after the insertion of S Steiner points.*



(a) (b) `Triangle`(1.4), $S = 50$ (c) `Triangle`(1.4), $S = 1000$

**Figure 4:** *Termination problem is shown for $\alpha = 35^\circ$ on a simple data set: a pair of points at three unit distance from each other inside a box of side length 100 units. Bad triangles are shaded (red). (a) Delaunay triangulation of the input. (b) and (c) Evolving triangulation using the circumcenter insertion. The refinement process does not terminate, hence the execution is interrupted after the insertion of S Steiner points.*

circumcenter refinement. This can be fixed by changing the bad triangle handling order.

*With or without (small input angle) constraints.* Prior to our work termination problem was addressed in the context of small input angles [MPW05, PW05, She00]. It is important to note that in general the termination problem exists regardless of the simplicity of the input domain. See Figure 4, where the input consists of a pair of points enclosed by a large box. In general, for any input domain, it is common to observe overrefinement occurring in regions far away from the input features whenever $\alpha$ is large. In this paper, we do not limit ourselves to certain type of constraints and rather address the termination problem in general. Our analysis in Section 4 focuses on point sets. However, this work can be easily coupled with the previous work [She02, MPW05, Üng04] that consider general planar straight line graphs. Our experiments, on data sets that are planar straight line graphs with small angles verify this.

### 2.2. Mesh Size

The output size is crucial for the efficiency of the methods using these meshes in various applications. Also, the smaller the number of Steiner points the faster the refinement algo-

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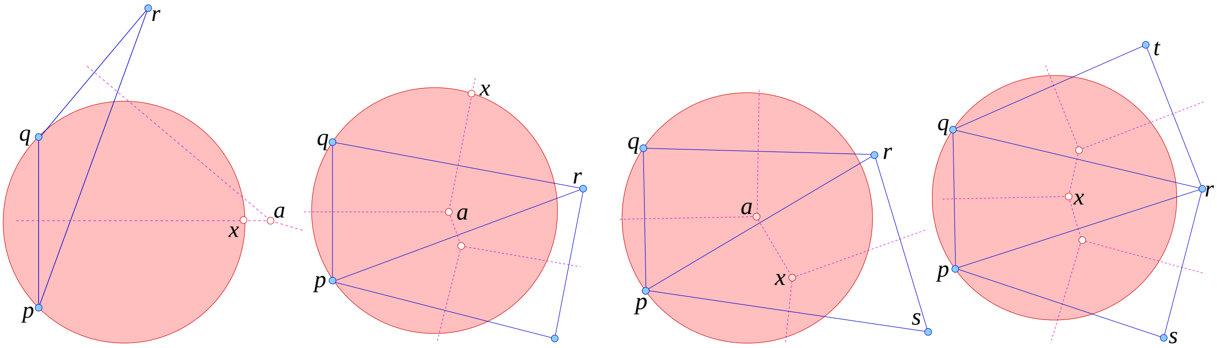


**Figure 5:** *Find a point x inside the (shaded disk) petal of pq that is furthest away from all existing vertices. Such a point can be on a Voronoi edge (left two) or a Voronoi vertex (right two).* TYPE I*: on the dual of pq;* TYPE II*: on a Voronoi edge other than the dual of pq;* TYPE III*: a circumcenter other than that of pqr;* TYPE IV*: the cirumcenter of pqr.*

rithm will be (if the Steiner point computation time is kept the same). Simpson [Sim06] recently showed empirical evidence that standard Delaunay refined meshes (through circumcenter insertion) are roughly twice the size of the optimal meshes for an application he calls function approximation. Our results here complement this study as we get a size improvement of roughly factor two compared to the previous Delaunay refinement algorithms.

## 3. LOCALLY OPTIMAL STEINER POINTS

**Definition 1** Given an ordered pair of points $(p,q)$ in the input domain. Consider the circle that goes through $p$ and $q$, of radius $2\sin\alpha|pq|$ whose center is on the right side of the directed edge $pq$. The disk bounded by this circle is called the *petal* of $pq$, denoted by $petal(pq)$.

Note that every edge has two petals (of interest), unless both its endpoints are on the boundary. In a triangulation, petals (of interest) of every edge $pq$ must include another vertex. Otherwise, there would be a bad triangle in the triangulation incident to $p$ and $q$. HarPeled and Üngör [HPÜ05] proved that the triangulation of a points set is good if and only if petal of every pair of points contains another point. In the following algorithm, we suggest to pick a Steiner point inside empty petals furthest away from all existing vertices.

**Algorithm 1**

Compute the Delaunay triangulation of the input
**while** $\exists$ a bad triangle $pqr$ with shortest edge $pq$
  insert a point $x \in petal(pq)$ of its shortest edge $pq$
    which is furthest from all existing vertices

This point is either a Voronoi vertex (but not necessarily the circumcenter of $pqr$) or on a Voronoi edge. We classify these points into four different types in order to relate them with the existing refinement strategies. This classification also helps us presenting a theoretical and experimental assessment of our method. Let $pqr$ be a bad triangle whose shortest edge is $pq$. Then, the point inside the petal of $pq$ furthest away from all existing vertices is one of the following types.

- TYPE I. The intersection of the Voronoi dual of $pq$ and the boundary of the petal of $pq$. This case happens when the circumcenter of $pqr$ is outside the petal of $pq$. This type is nothing but the offcenters described in [Üng04].
- TYPE II. A point on a Voronoi edge other than the dual of the shortest edge $pq$. This is a new type of Steiner point for Delaunay refinement and should be seen as an extension of the offcenter idea to the triangles that are almost good (circumcenter of $pqr$ is inside the petal of $pq$) .
- TYPE III. The circumcenter of a nearby triangle to $pqr$. Such a triangle must be acute. So, this type is a sink circumcenter. However, it is not necessarily the sink of the considered bad triangle $pqr$ (as is the case in Figure 5). Hence, our use of sinks is somewhat different than the original sink insertion algorithm of Edelsbrunner and Guoy [EG01].
- TYPE IV The circumcenter of the bad triangle $pqr$.

We compute the location (and also the type) of the Steiner point simply by doing a local search (breadth-first-search) on the Voronoi graph. The angles of the visited triangles provide guidance in this search. For instance, if the smallest angle of the bad triangle is less than $\alpha/2$, then we know for sure that the locally optimal Steiner point is of TYPE I (an offcenter). In general, it is better to start the search visiting the triangle opposite to the largest angle of the bad triangle. We elaborate more on the implementation details of the algorithm in Section 6.

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## 4. ANALYSIS

The analysis of the Delaunay refinement algorithms relies on lower bounds on the local feature size of the Steiner points. It is relatively easy to extend the analysis of the standard refinement techniques for our algorithms, since the local feature size of the Steiner points we use are not any smaller than that of the circumcenters. See for instance [Üng04] for an analysis of the refinement that uses Type I and Type IV Steiner points. Indeed, one might expect an improvement on the angle bound. Such an improvement, however, seems difficult to prove and left as an open problem. Here, we provide some theoretical evidence on why our strategy works for large values of threshold angle $\alpha$. For this purpose, the following lemma should be complemented with the experimental results on the percentage use of each type of Steiner points (presented in Section 6).

We further classify the Type II vertices for the sake of the analysis. Consider a Steiner point that is on the dual edge of $qr$ of a bad triangle with shortest edge $pq$. This TYPE II Steiner point is called a TYPE II (A) if $\angle qpr \geq \pi/2$, a TYPE II (B) if $\angle pqr \leq \angle qpr < \pi/2$, and a TYPE II (C) otherwise.

**Lemma 1** Given a point set $\Omega$ and a desired minimum angle $\alpha$ as input, the ALGORITHM 1 does not create any feature shorter than the existing ones while inserting a vertex $x$ of type

(a) TYPE I Steiner points if $\alpha \leq \pi/3 = 60^\circ$.
(b) TYPE II (A) Steiner points if $\alpha \leq \pi/4 = 45^\circ$.
(c) TYPE II (B) Steiner points if $\alpha \leq \pi/5 = 36^\circ$.
(d) TYPE II (C), TYPE III and TYPE IV Steiner points if $\alpha \leq \pi/6 = 30^\circ$.

*Proof* Let $pqr$ be a bad triangle with shortest edge $pq$.
($a$) Since $x$ is on the Voronoi edge of $pq$, its nearest neighbors among the existing vertices are $p$ and $q$. Observing that $|xp| < |pq|$ if and only if $\angle pxq > \pi/3$ completes this part of the proof.
($b$) Assume, without loss of generality, that $x$ is on the Voronoi dual of $qr$. So, $\angle qpr$ is a non-acute angle. Let $l$ be the line the line orthogonal to $pq$ and goes through $p$. (Figure 6 (top).) Let $y$ be the other (than $p$) intersection of line $l$ and $\partial petal(pq)$. Without loss of generality assume that $petal(pq)$ is unit disk. Then, the length of the arc $pq$ is $2\alpha$ (in radians). Let $x'$ be the midpoint of the arc $yq$. This means, the length of the arc $x'q$ is $\pi/2$. Hence, $\alpha \leq \pi/4$ if and only if $pq \leq x'q$. Note that there is no point below the line $l$ and outside the $petal(pq)$ that is closer to $x'$ than $q$. So, $|x'q| \leq |xq|$. Then, we conclude that $|xq| \geq |pq|$ if $\alpha \leq \pi/4$.
($c$) This part of the proof is similar to ($b$). Assume, without loss of generality, that $x$ is on the Voronoi dual of $qr$. So, $\angle pqr \leq \angle qpr < \pi/2$. Let $l$ be the line the line orthogonal to $pq$ and goes through the midpoint of $pq$. (Figure 6 (bottom).) Let $y$ be the intersection point of line $l$ and $\partial petal(pq)$ that is furthest from $pq$. Without loss of generality assume that $petal(pq)$ is unit disk. Then, the length of the arc $pq$ is $2\alpha$ (in radians) and the length of the arc $yq$ is $\pi - \alpha$. Let $x'$ be the midpoint of the arc $yq$. This means, the length of the arc $x'q$ is $\pi/2 - \alpha/2$. Hence, $\alpha \leq \pi/5$ if and only if $pq \leq x'q$. Note that there is no point below the line $l$ and outside the $petal(pq)$ that is closer to $x'$ than $q$. So, $|x'q| \leq |xq|$. Then, we conclude that $|xq| \geq |pq|$ if $\alpha \leq \pi/5$.
($d$) Straightforward to show. □

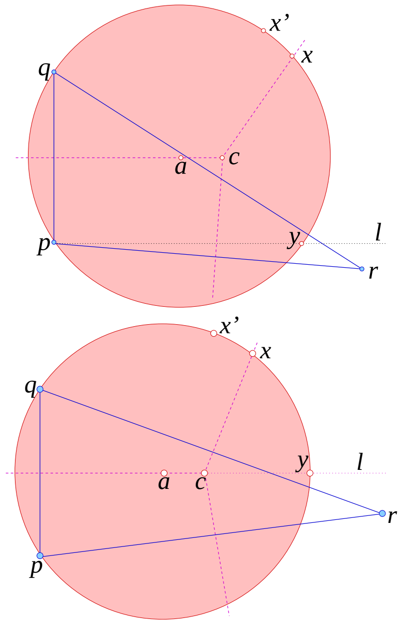


**Figure 6:** TYPE II (A) AND (B)

Lemma 1 suggests that if we could show that the ALGORITHM 1 only uses, say TYPE I and TYPE II (A) Steiner points, then it would (provably) compute triangulations with minimum angle of $45^\circ$. Unfortunately, such a premise does not seem plausible. Our experiments (presented in Section 6) show that all four types are employed by Algorithm 1 in different amounts.

The following theorem follows from the above Lemma. Its proof is similar to the results in traditional Delaunay refinement [Rup93, She02, Üng04] and hence omitted here.

**Theorem 1** Given a planar straight line graph $\Omega$ and a desired minimum angle $\alpha \leq 30^\circ$ ALGORITHM 1 terminates with a correct output.

*Time Complexity.* A straightforward running time analysis leads to a quadratic time complexity bound for ALGORITHM 1 as is the case for most other Delaunay refinement algorithms. However, this bound can be improved to $O(n \log n + m)$, where $n$ is the input size and $m$ is the output size, by using a technique recently introduced [HPÜ05]. This technique employs a balanced quadtree as a data structure and takes advantage of the locality of the Steiner points with respect to the shortest edge of bad triangles.

## 5 . RELOCATION and REFINEMENT

Based on our analysis (Section 4) and experiments with Algorithm 1, we observed that the refinement process introduces shorter edges than existing ones usually when a bad triangle is almost good and also the neighbor triangles are good or almost good. This suggests that it might be easy to fix such bad triangles by a local smoothing strategy. While one might explore various powerful smoothing strategies [ABE97] in these cases, we opt for simplicity and efficiency. We first recall couple of definitions and then describe a simple adjustment to our algorithm. The *star* of a vertex $a$ consists of all triangles that contain $a$. The *link* of $a$, then, consists of all edges of triangles in the star that are disjoint from $a$. A vertex is said to be *free* if it was inserted by the refinement algorithm as a Steiner point. Input vertices are not free and never relocated. For each bad triangle, we first (one-at-a-time) attempt to relocate its free vertices. If one of its free vertices find a new location so that all the triangles in its (new) star become good, we perform the relocation. Otherwise, we proceed with a new Steiner point insertion as described in ALGORITHM 1.

**Algorithm 2**

Compute the Delaunay triangulation ($DelTri$) of the input
Let $P$ denote the maintained point set
**while** $\exists$ a bad triangle $pqr$ in $DelTri(P)$
  relocated := FALSE
  **for** each free vertex $a$ of $pqr$
    **if** $\mathcal{K} = \bigcap_{xy \in link(a)} petal(xy) \neq \emptyset$
      **and** $\exists b \in \mathcal{K}$ such that all triangles of $star(b)$
        in the $DelTri(P \cup \{b\} - \{a\})$ are good
    **then**
      delete $a$; insert $b$; relocated:=TRUE; **break**;
  **endfor**
  **if** relocated == FALSE **then**
    insert a point $x \in petal(pq)$ of its shortest edge $pq$
      which is furthest from all existing vertices
**endwhile**

The standard analysis techniques (for proving termination and output size optimality) that are used for the previous Delaunay refinement algorithms do not apply for ALGORITHM 2 due to the point relocation step. While proving an improved bound for it is left open, we can match the earlier bounds by a simple modification to ALGORITHM 2: apply the relocation only when the the constraint angle is larger than 30. This modification keeps ALGORITHM 2 still effective for large constraint angles and provably good for small constraint angles.

## 6. EXPERIMENTS

We implemented the proposed algorithms and run experiments on various data sets and point distributions. Figures 10 and 11 and Table 6.3 summarizes our experimental study on ALGORITHM 1 and ALGORITHM 2. Performance plots are similar for various other data sets. We should emphasize that while as an illustration we present several sample output triangulations in Figures 1, 8, and 9, performance plots in Figures 10 and 11 provide much more information regarding the performance of the two algorithms.

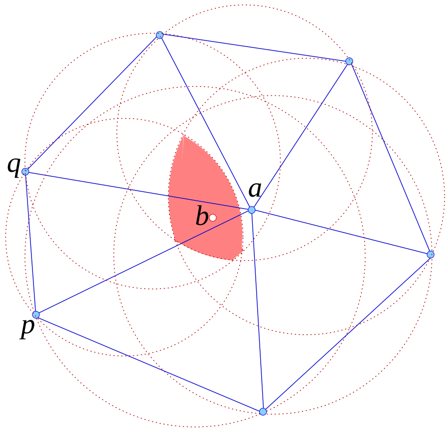


**Figure 7:** *Relocating a free vertex of a bad triangle $pqr$ to the intersection of the petals of the link of $a = r$.*

### 6.1. Implementation

Our implementation is a fairly modest modification of the `Triangle` software. Here, we explain the crucial changes. For computing the TYPE II Steiner points, we have implemented a primitive that computes the intersection(s) of a ray and a circle. This computation is slightly more expensive than computing circumcenters and is common to graphics software [MST89]. A similar primitive that computes the intersections of two circles is implemented for the relocation step of ALGORITHM 2. In this step, however we avoid computing the exact intersection of all petals on the link. Instead, we use a simple sampling strategy which proves effective and efficient. For each pair of petals, we pick a sample of points on the line segment connecting the intersection points of the two petals. We enumerate the petal pairs starting from those that are furthest from each other on the link. If there exist any petal pair with no intersection, then we terminate the process and conclude that there is no relocation point. Otherwise, we test for each sampling point $b$ whether all triangles of $star(b)$ in the Delaunay triangulation of the set $(P \cup \{b\} - \{a\})$ are of good quality, where $P$ is the evolving point set and $a$ is the point to be relocated. We always maintain the Delaunay property of the triangulation. Alternative strategies for point relocation can be explored. We choose this one for its simplicity end efficiency.

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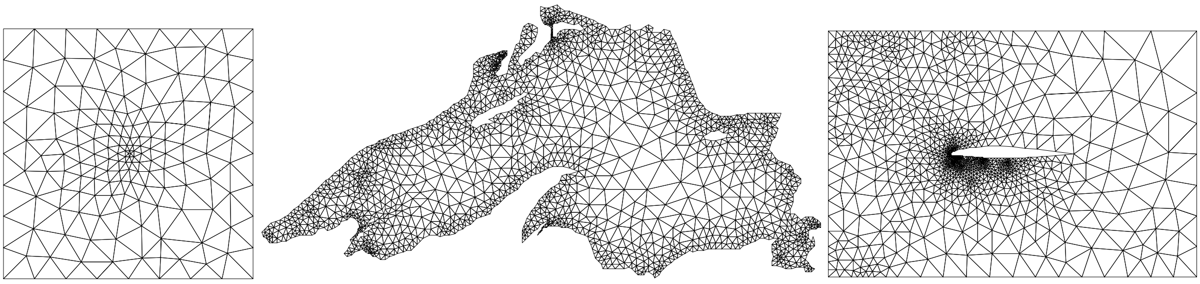

$\alpha = 42^\circ$ $\alpha = 41^\circ$ $\alpha = 41.5^\circ$

**Figure 8:** *Output of the* ALGORITHM 2 *for various data sets for various large* $\alpha$ *values.*

### 6.2. Data Sets.

We ran our experiments on various data sets including the following:

1. *Lake Superior* consists of 522 points, 522 segments some of which meet at small angles, and 7 holes.
2. *Boxed Point Pair* consists of six points two of which are located unit distance from each other and at the center of a square of side length 100 units.
3. *Boeing* consists of 30 points and 30 segments and a hole modeling an airplane wing. The sampling around the tip of the wing is very fine (to allow accurate simulations).
4. *Random Points* consists of $1,000$ points spread uniformly at random inside a square box.
5. *Turkey* consists of 216 points and 216 segments and two components.

### 6.3. Performance Comparison

We compare our algorithms with the previous Delaunay refinement implementations on three performance measures.

*Angle threshold.* Plots in Figure 11 show that that the original Delaunay refinement algorithm is impotent for constraint angles larger than $34^\circ$. The offcenter insertion algorithm of Üngör has already extended this cut-off to $35^\circ$. On the other hand, our ALGORITHM 1 terminates with correct output for constraint angles up to $38.5^\circ$. Finally, our ALGORITHM 2 which is a simple extension of the first algorithm works for constraint angles up to $42^\circ$.

*Output size.* The number of triangles in a triangulation is a simple linear function of the number of points in it. Hence, the plots in Figure 11 reflect on the number of triangles in the output. We observed significant improvement on the output size of the two refinement algorithms proposed here. This improvement is particularly impressive when the threshold angle is large. (See Figure 9 also.)

*Running time.* The primitives we use for computing the proposed locally optimal Steiner points is slightly more expensive than those used for computing circumcenters. However, we insert fewer Steiner points. Overall, ALGORITHM 1 runs faster than the previous algorithms, where as ALGORITHM 2 has comparable running time. (See Figure 10.) After the optimization of our code, we expect more significant speed up on both algorithms.

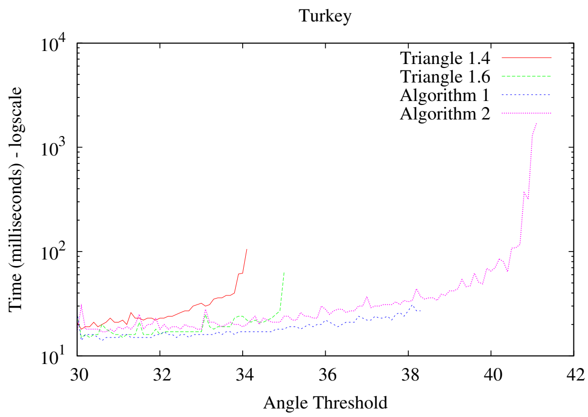


**Figure 10:** *Plot of the running time vs. the constraint angle* $\alpha$ *for the Turkey data set shown in Figure 1.*

## 7. DISCUSSIONS

The termination and size complexity bounds given for the previous Delaunay refinement apply for our algorithms, for constraint angles up to $30^\circ$, as we are more cautious in introducing short features. It would be interesting to prove the same theoretical (termination and size-optimality) bounds for $\alpha > 30^\circ$. It would be also interesting to further improve the practical performance angle bounds say for $\alpha \geq 45^\circ$. This would imply generating non-obtuse/acute angle triangulations. Ours is a first Delaunay refinement result breaking the constraint angle barrier of $34^\circ$, which survived over ten years. In achieving this, its important to note that we kept the simplicity of the algorithm. This in turn, enabled us to design an efficient and effective implementation. Alternative (more powerful but perhaps more expensive) mesh optimization algorithms can be integrated within our framework.

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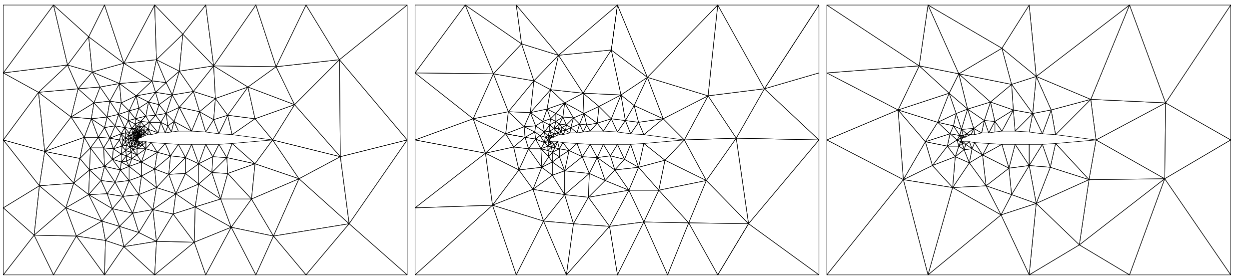

`Triangle`(1.4) (circumcenter) `Triangle`(1.6) (offcenter) ALGORITHM 2

**Figure 9:** *Output size comparison on the Boeing data set. For* $\alpha = 30^\circ$*, the new algorithm inserts* 62 *Steiner points, almost half as many as the* 118 *Steiner points inserted by the offcenter algorithm which is in turn half as many as the* 236 *Steiner points used by the circumcenter insertion algorithm.*

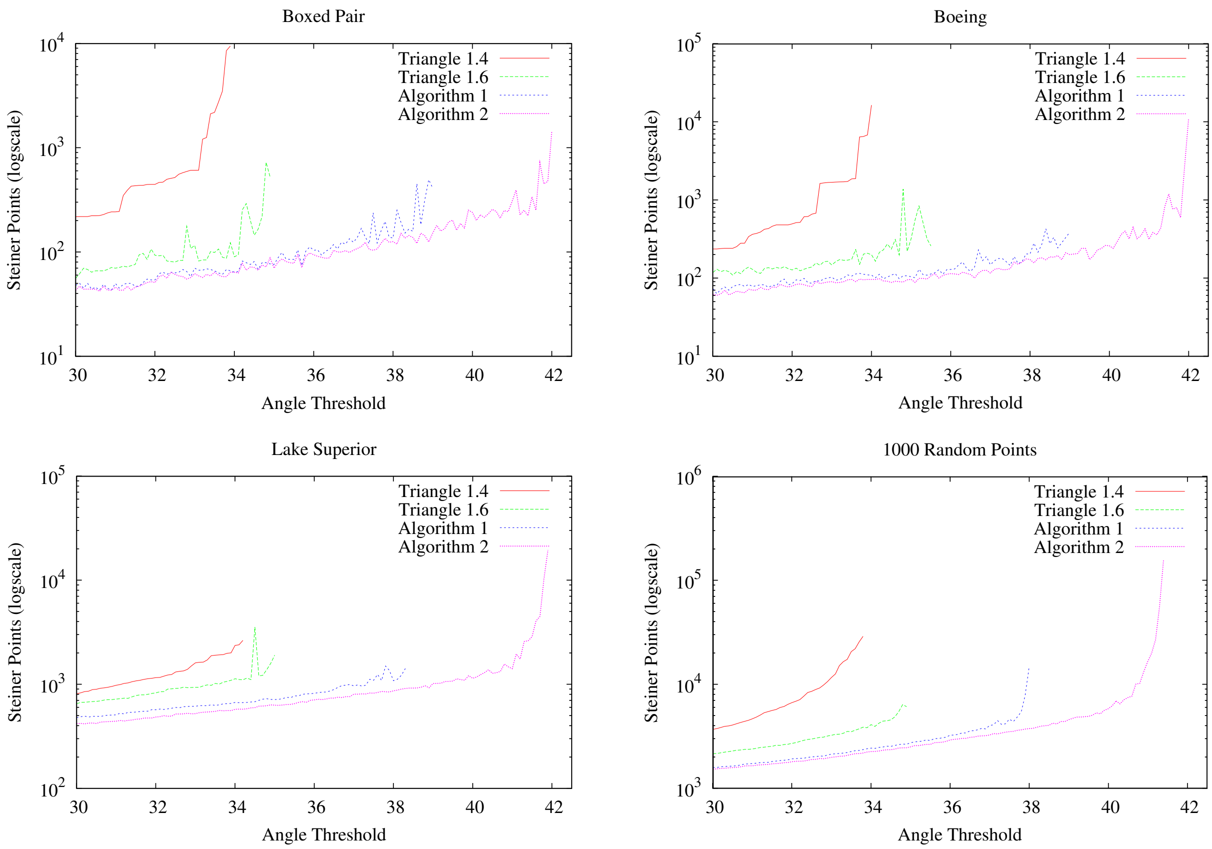


**Figure 11:** *Plot of the number of Steiner points vs. the constraint angle* $\alpha$*.*

The standard Delaunay refinement algorithms are prone to significant round-off errors, e.g., computation of the circumcenter of a triangle with a very large angle. Since we limit ourselves to petal regions, we avoid such round-off errors. Hence, the algorithms and implementation presented in this work are numerically stable.

Delaunay refinement is a popular technique for computing surface triangulations also [Dey06]. We foresee that our algorithms can be easily extended for computing high quality and small size triangulations of two manifolds. Extension of method to three dimensions is also a natural research direction and is currently under study.

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| | Circumcenter | Offcenter | | Algorithm 1 | | | | Algorithm 2 | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Data Set $\alpha$ | TypeIV | TypeI | TypeIV | TypeI | TypeII | TypeIII | TypeIV | TypeI | TypeII | TypeIII | TypeIV | Relocation |
| Superior 30 | 803 | 182 | 455 | 136 | 152 | 176 | 7 | 145 | 133 | 127 | 6 | 76 |
| Superior 34 | 2350 | 323 | 787 | 185 | 228 | 237 | 10 | 193 | 203 | 164 | 5 | 79 |
| Superior 38 | $\infty$ | $\infty$ | $\infty$ | 269 | 379 | 381 | 24 | 254 | 316 | 253 | 14 | 101 |
| Superior 41 | $\infty$ | $\infty$ | $\infty$ | $\infty$ | $\infty$ | $\infty$ | $\infty$ | 384 | 567 | 331 | 68 | 206 |
| Boeing 30 | 233 | 59 | 103 | 34 | 42 | 12 | 0 | 39 | 29 | 8 | 0 | 7 |
| Boeing 34 | 16309 | 88 | 154 | 50 | 57 | 20 | 3 | 48 | 55 | 12 | 1 | 5 |
| Boeing 41 | $\infty$ | $\infty$ | $\infty$ | $\infty$ | $\infty$ | $\infty$ | $\infty$ | 95 | 152 | 47 | 8 | 47 |
| Random 30 | 3519 | 593 | 1380 | 445 | 426 | 508 | 30 | 446 | 421 | 458 | 24 | 58 |
| Random 34 | $\infty$ | 1076 | 2770 | 677 | 666 | 802 | 70 | 675 | 644 | 672 | 43 | 141 |
| Random 41 | $\infty$ | $\infty$ | $\infty$ | $\infty$ | $\infty$ | $\infty$ | $\infty$ | 3853 | 5857 | 5038 | 1344 | 2744 |

**Table 1:** *Number of different type of Steiner points used by four different Delaunay refinement algorithms.*

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