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A Geometric Construction of Coordinates for Convex Polyhedra using Polar Duals

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Abstract

A fundamental problem in geometry processing is that of expressing a point inside a convex polyhedron as a combination of the vertices of the polyhedron. Instances of this problem arise often in mesh parameterization and 3D deformation. A related problem is to express a vector lying in a convex cone as a non-negative combination of edge rays of this cone. This problem also arises in many applications such as planar graph embedding and spherical parameterization. In this paper, we present a unified geometric construction *for building these weighted combinations using the notion of polar duals. We show that our method yields a simple geometric construction for Wachspress's barycentric coordinates, as well as for constructing Colin de Verdière matrices from convex polyhedra—a critical step in Lovasz's method with applications to parameterizations.*

Categories and Subject Descriptors (according to ACM CCS): I.3.5 [Computer Graphics]: Geometric algorithms, languages and systems

1. Introduction

Expressing a point as a weighted combination of other points is a fundamental problem in geometry processing. For example, free-form deformations [SP86, Coq90, MJ96, KO03] represent points in space as affine combinations of vertices of a control mesh. When the user modifies the position of the vertices of the control mesh, points are deformed using those affine combinations. Other applications such as boundary interpolation [JSW05] use these weighted combinations to extend boundary functions to the interior of the object. Mesh parameterization [HG00, DMA02, KLS03, KO03, FH05] also requires points as weighted combinations of other points, sometimes with the added restriction that the weights are positive. In particular, given a convex polytope P with vertices v_i, we wish to write an interior point x as a convex combination

$$\sum_i b_i v_i = x \quad \text{where} \quad b_i \geq 0 \quad \text{and} \quad \sum_i b_i = 1.$$

The weights b_i are often called the *barycentric coordinates* of point x on polytope P. Constructing such coordinates is often not a trivial task.

Background on Barycentric Coordinates If P is a simplex (e.g., a triangle in $2D$ or a tetrahedra in $3D$), the barycentric coordinates b_i for an interior point x can be intuitively expressed as the ratio of the *volume* of a simplex, formed by x and the opposite face of v_i, divided by the volume of P (see figure 3 for an example on a triangle). However, this simple geometric construction can not be extended beyond simplicies since the opposite face to a vertex is no longer well-defined.

As an alternative, Wachspress [Wac75] proposed a construction of barycentric coordinates for convex polygons in $2D$, which was later refined by [LD89] and [MLBD02]. The coordinates are well behaved in the interior of the polygons and have a direct geometric interpretation in $2D$ as ratios of areas. Unfortunately, the generalization of Wachspress's construction to $3D$ and higher dimensions [War96, WSHD04] is harder to understand geometrically due to the presence of non-simplicial vertices (i.e., vertices whose valence is larger than the dimensionality of the embedded space). Other researchers have also proposed alternative forms of barycentric coordinates that can be extended onto non-convex polytopes in $2D$ [MD03, Flo03, Hor04] and $3D$ [FKR05, JSW05].

Vector as Convex Sum of Vectors A problem closely related to building weighted combination of points is express-

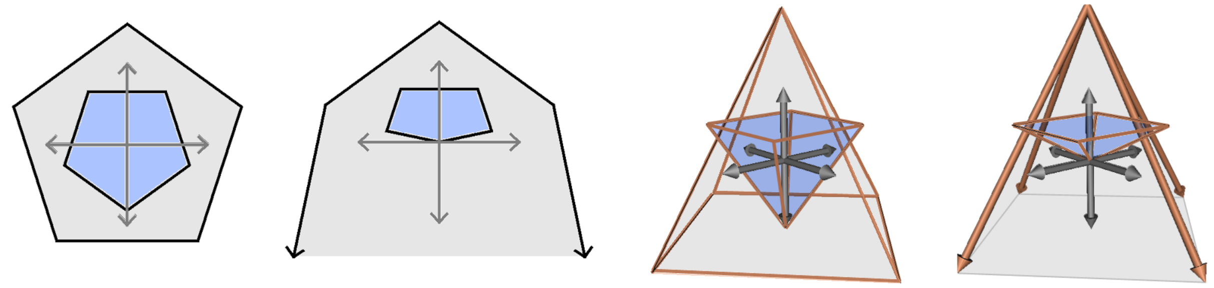

Figure 1: *Bounded and unbounded convex polyhedra in* 2 *and* 3 *dimensions (shaded gray) with their corresponding polar duals (shaded blue).*

ing a vector as a weighted combination of containing vectors. In particular, given a convex polyhedral cone bounded by a set of one-dimensional rays pointing in the directions r_j, we wish to write a vector v lying in the cone as a non-negative linear combination of these vectors:

$$\sum_j c_j r_j = v \quad \text{where} \quad c_j \geq 0.$$

To the best of our knowledge, this vector problem has never been considered in isolation. However, a solution to this problem is proposed as part of a construction by Lovasz [Lov01] for building a special type of weighted adjacency matrix called a Colin de Verdière matrix (CdV) from a convex polyhedron. These matrices play a key role in planar graph embedding and spherical parameterization [GGS03]. Unfortunately, Lovasz's construction is rather abstract and lacks a simple geometric interpretation.

Contributions In this paper we present a single, simple geometric construction for solving both weighted combination problems. The construction is based on applying Stokes' Theorem [Fle77] to the polar dual [Gru67] of a convex polyhedron. Our geometric construction offers the following insights:

- Applied to a bounded convex polyhedron, the construction yields a simple, intuitive formula for Wachspress coordinates as the *ratios of volumes* of certain dual polyhedra, nicely extending the usual notion of barycentric coordinates in a simplex.
- Applied to a convex polyhedral cone, this method provides an explicit construction for expressing an interior vector as a non-negative combination of the edge rays of the cone.
- This vector construction also yields a simple geometric explanation of Lovasz's method for building CdV matrices from a convex polyhedron.

2. Dual-based Geometric Construction of Coordinates

To simplify our exposition, we present our construction in 3D (including both 2D and 3D examples). However, our geometric construction extends to arbitrary dimension without difficulty. A convex polyhedron P is the intersection of a finite set of half-spaces in R^3. Such a polyhedron may be bounded or unbounded as shown in figure 1. In the unbounded case, the polyhedron (or polygon) includes unbounded edges that form rays.

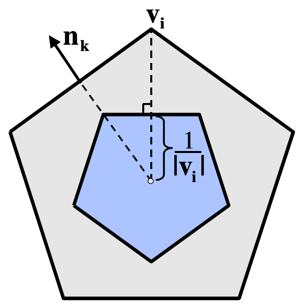

Figure 2: *The dual (shaded blue) of a convex pentagon (shaded grey) with various geometric properties highlighted.*

Definitions The *polar dual* of a convex polyhedron that contains the origin is itself a convex polyhedron of the form

$$d[P] = \{y | y \cdot z \leq 1 \; \forall z \in P\}.$$

Figure 1 shows the polar duals of various convex polygons in 2D and polyhedra in 3D. Observe that if P is bounded, the dual $d[P]$ contains the origin in its interior. On the other hand, if the polyhedron P is unbounded, the origin lies on the boundary of $d[P]$.

Polar duals have a number of useful properties [Gru67]. For our purposes, the most interesting property is that each vertex v_i of P corresponds to a face of $d[P]$ whose outward normal is v_i and whose distance from the origin is $\frac{1}{|v_i|}$. Conversely, each face of P with outward normal n_k corresponds to a vertex of $d[P]$ at location $\frac{n_k}{n_k \cdot v}$ where v is any vertex on the face: this face and its associated vertex are said to be *dual* to each other. Figure 2 illustrates these properties.

Unbounded Polyhedra Note that if the polyhedron P is bounded, this mapping between vertices of P and faces of $d[P]$ is one-to-one. However, when P is *unbounded*, the vertices of P are dual to only a subset of the faces of $d[P]$. In this case, the remaining faces of $d[P]$ contain the origin and are dual to the *unbounded rays* r_j of P. These rays are unbounded edges where two bounding planes met. Specifically, the rays r_j (treated as vectors along the ray) are the outward normals to these dual faces. The unbounded pentagon in figure 1 and its polar dual illustrate this point.

Main Result We now state our fundamental theorem relating the vertices v_i and the rays r_j of P.

Theorem: Consider a convex polyhedron P containing the origin with vertices v_i and rays r_j. Then,

$$\sum_i \alpha_i \frac{v_i}{|v_i|} + \sum_j \beta_j \frac{r_j}{|r_j|} = 0 \qquad (1)$$

where α_i is the area of the face dual to v_i in $d[P]$ and β_j is the area of the face dual to r_j in $d[P]$.

□ *Proof*: According to Stokes' theorem [Fle77], the integral of the outwards unit normal over any closed surface is zero. When applied to $d[P]$, the integral becomes the sum of the area of each planar face multiplied by the corresponding outward unit normal, which is exactly the left-hand side of the equation. ■

In the remaining parts of the paper, we consider two applications of this theorem. In the next section, we express a point in the interior of a convex polyhedron as a convex combination of its vertices yielding a geometric interpretation of Wachspress's coordinates in arbitrary dimension. In the subsequent section, we express a vector lying in a convex polyhedral cone as a linear combination of the rays of the cone and utilize this construction to deduce a geometric formulation of Lovasz's method for building a CdV matrix from a convex polyhedron.

3. Points as Convex Combination of Vertices

A common problem in applications such as parameterization and deformation is to express a point x on the interior of convex polyhedron P as a convex combination of the vertices v_i of P. Given $x \in P$, our task is to find a set of non-negative coordinates b_i (depending on x) such that x satisfies

$$\sum_i b_i v_i = x.$$

In solving this problem, a standard approach is to compute a set of associated weights ω_i that satisfy the relation

$$\sum_i \omega_i (v_i - x) = 0.$$

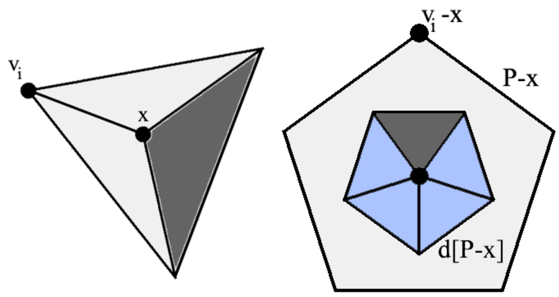

Figure 3: *The simplicial formula for b_i is the ratio of the shaded area to the area of the triangle (left), but does not extend beyond simplicies. Barycentric coordinates for a pentagon via partitioning of its dual (right). The coordinate b_i for v_i is the ratio of the dark area to the entire shaded dual.*

and then define the coordinates b_i to be of the form $\omega_i / \sum_i \omega_i$.

At this point, we can now apply equation 1 to construct suitable weights ω_i. To construct the appropriate polar dual, we translate P such that the point x is now at the origin. The vertices of the translated polyhedron $P - x$ are now $v_i - x$. Since $P - x$ is bounded and has no rays, equation 1 yields weights of the form

$$\omega_i = \frac{\alpha_i}{|v_i - x|}. \qquad (2)$$

where the α_i are the areas of the faces dual to the vertices $v_i - x$.

3.1. Geometric Interpretation

Like the simplicial formula from figure 3 (left), this expression has an elegant geometric interpretation in terms of ratios of volumes. However, unlike the simplicial formula, our formula has a natural interpretation that extends to arbitrary polytopes. To illustrate this point, consider a convex polytope P with vertex v_i (see figure 3, right). The face dual to $v_i - x$ forms a pyramid whose apex is the origin. The volume of this dual pyramid is proportional to the area of its base α_i times its height. Now, the height of the pyramid is simply the distance of the face dual to $v_i - x$ from the origin, that is $\frac{1}{|v_i - x|}$. Therefore, the weight ω_i is proportional to the volume of the pyramid dual to $v_i - x$.

Since the pyramids dual to the vertices $v_i - x$ form a disjoint partition of $d[P - x]$, the sum of the weights $\sum_i \omega_i$ is proportional to the volume of $d[P - x]$. Therefore, the coordinates b_i are simply the *ratio of the volume* of the pyramid dual to $v_i - x$ divided by the volume of the dual of $P - x$. Figure 3 shows an example of this construction applied to a pentagon compared with the simplicial volume-based formula.

3.2. Relation to Wachspress Coordinates

Here we show that the ω_i defined in equation 2 using the polar duals are in fact the weights associated with the classic Wachspress coordinates for convex polygons [MLBD02] and polyhedra [WSHD04].

In 2D, the weight ω_i has a simple explicit formula. Given a convex polygon with vertices v_i, let n_i be an outward normal to the edge (v_i, v_{i+1}). Now, the vertex $v_i - x$ is dual to the edge in $d[P-x]$ with endpoints $\frac{n_{i-1}}{n_{i-1}\cdot(v_i-x)}$ and $\frac{n_i}{n_i\cdot(v_i-x)}$. Therefore, the weight ω_i is the area of the triangle formed by this dual edge and the origin. This area can be expressed as a determinant of the form

$$\omega_i = \frac{Det[n_{i-1}, n_i]}{(n_{i-1}\cdot(v_i-x))(n_i\cdot(v_i-x))}.$$

If we chose length of the normal vector n_i to be the length of the edge (v_i, v_{i+1}), this formula reduces to the standard one given in [MLBD02]

$$\omega_i = \frac{Det[v_{i+1}-v_i, v_{i-1}-v_i]}{Det[v_{i-1}-x, v_i-x]Det[v_i-x, v_{i+1}-x]}.$$

3.3. Relation to Warren's Coordinates

In 3D, we can compute the weight ω_i associated with a vertex v_i of the polyhedron P as well. If $v_i - x$ has valence d, let $n_1, n_2, \ldots n_d$ be normals to the d faces containing v_i (enumerating in cyclic order). Each of the faces of $P - x$ is dual to a vertex of $d[P-x]$ that bounds the face dual to $v_i - x$. In particular, these vertices have the form $\frac{n_k}{n_k\cdot(v_i-x)}$.

Now, the weight ω_i is proportional to the volume of the pyramid spanned by d vectors emanating from the origin to these points. Specifically,

$$\omega_i = 3Vol[\frac{n_1}{n_1\cdot(v_i-x)}, \frac{n_2}{n_2\cdot(v_i-x)}, \ldots, \frac{n_d}{n_d\cdot(v_i-x)}]. \tag{3}$$

To compute this volume explicitly, we suggest triangulating the face dual to $v_i - x$ and using this 2D triangulation to induce a 3D tetrahedralization on the corresponding pyramid dual to $v_i - x$. Given a triangle with vertices $\frac{n_1}{n_1\cdot(v_i-x)}, \frac{n_2}{n_2\cdot(v_i-x)}, \frac{n_3}{n_3\cdot(v_i-x)}$, the corresponding tetrahedron has a volume

$$\frac{\frac{1}{6}Det[n_1, n_2, n_3]}{\prod_{k=1}^{3} n_k\cdot(v_i-x)}.$$

This determinant expression in conjunction with equation 3 agrees with the formulas for arbitrary valence vertices given in [WSHD04].

4. Vectors as Linear Combination of Edges

The previous section considered the problem of writing a point as a convex combination of enclosing vertices. In this section, we consider the related problem of writing a vector v as a non-negative linear combination of enclosing rays r_j. Compared to the point problem, the vector problem has received relatively little attention. Nonetheless, applications involving this type of problem often arise in $3D$ parameterization [HG00, DMA02, KLS03, KO03] where a point on a non-planar surface needs to be expresses as linear combination of neighboring points on the surface. Here we restrict our discussions to convex surfaces, and will explore an interesting link to CdV matrices for convex meshes.

4.1. Building Vector Coordinates

Problem Statement To formally state our problem, consider a convex polyhedral cone P formed by taking the intersection of d halfspaces whose bounding planes contain the origin (the polyhedral cone on the right of figure 1 is an example). This cone is bounded by d rays of the form $r_1, r_2, \ldots r_d$. Now, given a vector v in P, we wish to construct a set of non-negative weights μ_j and ω such that

$$\sum_j \mu_j r_j = \omega v. \tag{4}$$

Our Construction To apply equation 1, we translate P so that v lies at the origin. The translated polyhedron $P - v$ has a single vertex $-v$ and d rays r_j. According to equation 1

$$\sum_j \beta_j \frac{r_j}{|r_j|} = \alpha \frac{v}{|v|} \tag{5}$$

where α is the area of the face dual to $-v$ in $d[P-v]$ and the β_j are the areas of the faces dual to the rays r_j. Comparing equation 4 and 5, we obtain the desired weights

$$\mu_j = \frac{\beta_j}{|r_j|}, \quad \omega = \frac{\alpha}{|v|} \tag{6}$$

As observed in the previous section, the ω is simply the Wachspress weight associated with the vertex v.

To derive an explicit formula for areas β_j, we let the ray r_j be the intersection of two planes with outward normals n_{j-1} and n_j. The dual of this ray is a triangular face of $d[P-v]$ whose vertices are $\frac{-n_{j-1}}{n_{j-1}\cdot v}, \frac{-n_j}{n_j\cdot v}$ and the origin. Therefore, the area β_j is simply

$$\beta_j = \frac{\frac{1}{2}|n_{j-1}\times n_j|}{(n_{j-1}\cdot(v-x))(n_j\cdot(v-x))}$$

(Note that in 2D, the areas β_j are simply $\frac{|n_j|}{n_j\cdot(v-x)}$.)

4.2. Geometric Construction of CdV Matrices

Equation 4 and 6 present a geometrically simple method for expressing a vector inside a convex cone as a non-negative linear combination of the cone's rays. We can now apply this method to reformulate Lovasz's construction [Lov01] for building a weighted adjacency matrix corresponding to the edge graph induced by a convex polyhedron.

Lovasz's Construction Given a bounded convex polyhedron P containing the origin with n vertices $v_i \in V$ and edges $(v_i, v_j) \in E$, Lovasz constructs an $n \times n$ symmetric matrix M with the following properties:

- $M_{ij} = 0$ if $(v_i, v_j) \notin E$,
- $M_{ij} > 0$ if $(v_i, v_j) \in E$ and $i \neq j$,
- $MV = 0$ where V is a matrix whose i^{th} row is v_i.

Essentially, M is a weighted adjacency matrix associated with the genus zero graph (V, E) whose nullspace includes V. The importance of this construction lies in the fact that Lovasz then shows that M is a special type matrix known as a Colin de Verdière (CdV) matrix. Such matrices are guaranteed to have a specific spectral structure. In particular, a CdV matrix M associated with a genus zero graph has a single positive eigenvalue, three eigenvalues at zero and the remaining eigenvalue all being strictly negative. These spectral properties make M well-suited for 3D mesh filtering similar to that used in Fourier analysis [Tau95]. In particular, CdV matrices play a key role in spherical parameterization as discussed in [GGS03].

Revisiting the Construction Our goal in this section is to give a much simpler version of Lovasz's construction for M by applying the method of section 4. Given v_i, consider the polyhedral cone formed by those halfspaces of P that meet at v_i. The rays of this cone simply correspond to those edges (v_i, v_j) that are incident on v_i. So, we can apply the method of section 4 to express the vector $-v_i$ as a weighted linear combination of the vectors $v_j - v_i$, i.e;

$$\sum_{(v_i,v_j)\in E} \mu_{ij}(v_j - v_i) = \omega_i(-v_i) \tag{7}$$

where ω_i is the Wachspress weight associated with the vertex v_i and the weights μ_{ij} satisfy equation 6. Collecting the coefficient of v_i yields an equivalent equation of the form

$$\sum_{(v_i,v_j)\in E} \mu_{ij} v_j + (\omega_i - \sum_{(v_i,v_j)\in E} \mu_{ij}) v_i = 0. \tag{8}$$

Based on this equation, we can now define M directly in terms of μ_{ij} and ω_i, i.e;

$$\begin{aligned} M_{ij} &= \mu_{ij} \quad (i \neq j) \\ M_{ii} &= \omega_i - \sum_{(v_i,v_j)\in E} \mu_{ij}. \end{aligned}$$

Note that $M_{ij} > 0$ for $(v_i, v_j) \in E$ since $\mu_{ij} > 0$, M_{ij} is symmetric since $\mu_{ij} = \mu_{ji}$ and $MV = 0$ by equation 8.

In 2D, the entries of M have a simple expression in terms of areas similar to that of [MLBD02]. If P is a convex polygon with vertices $v_1, v_2, \ldots, v_n$, M is a symmetric tridiagonal matrix whose i^{th} row has non-zero entries of the form

$$\begin{aligned} M_{i,i-1} &= \frac{1}{Det[v_{i-1}, v_i]} \\ M_{ii} &= \frac{-Det[v_{i-1}, v_{i+1}]}{Det[v_{i-1}, v_i] Det[v_i, v_{i+1}]}, \\ M_{i,i+1} &= \frac{1}{Det[v_i, v_{i+1}]}. \end{aligned}$$

Geometric Interpretation Observe that Lovasz's construction has an elegant interpretation in terms of polar duals. Given a convex polyhedron P, one computes the polar dual $d[P]$ and partitions this dual into pyramids with respect to the origin (as shown in figure 3 right). Each external face of $d[P]$ has outward normal v_i. Each internal face in the partition has a normal parallel to an edge (v_i, v_j) of P. To compute the ith row of M, one simply applies Stokes' theorem to the pyramid dual to v_i. Since the weights μ_{ij} and μ_{ji} depend only on the length of the edge (v_i, v_j) and the area of the internal face shared by the pyramids dual to v_i and v_j, the resulting matrix M is symmetric.

5. Conclusions

Using Stokes' theorem and polar duals of convex polyhedra, we have constructed a method for representing points in terms of vertices of a bounded convex polyhedron and vectors in terms of vectors corresponding to edges of an unbounded convex polyhedron. We then applied this technique to generating barycentric coordinates for convex polytopes as ratios of volumes. Our vector construction yielded a simple geometric derivation for Lovasz's method for constructing CdV matrices.

The heightened intuition presented in our paper of the geometry at play in basic graphics tools such as barycentric coordinates suggests both theoretical and practical consequences. It may first have interesting ramifications in the study of parameterizations thanks to the connection with CdV matrices. But a number of applications could also directly benefit, such as ray interpolation, or light field radiance interpolation (where a 5D sampled field needs proper local reconstruction) for rendering purposes. We also plan to explore the usefulness of this construction to the definition of basis function for discrete differential forms on arbitrary meshes [DKT05] as used in Geometry Processing, Electro-Magnetism, and Mechanics.

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