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Modeling with Simplicial Diffeomorphisms

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Abstract

In this paper we introduce a new framework for geometric modeling that combines implicit and parametric surface representations with volumetric warpings. The framework is based on dynamic, adaptive simplicial decompositions of the embedding space and conforming piecewise diffeomorphisms.

Categories and Subject Descriptors (according to ACM CCS): I.3.5 [Computer Graphics]: Computational Geometry and Object Modeling

1. Introduction

The geometry of 3D shapes is traditionally described eiher in parametric or implicit form. These two shape representations are complementary in many aspects. For example, the parametric representation gives a direct way to enumerate points of an object and is well suited for visualization. The implicit representation provides a way to evaluate the relation of points in the ambient space with an object and can be used for interference analysis.

In that respect, it would be desirable that both forms could be employed to describe objects in a modeling system. Some basic shapes, such as spheres, cylinders, etc, possess both parametric and implicit representations. This fact is exploited in many computational geometry algorithms.

However, complex objects usually don't have a dual parametric and implicit representation. Very often, they don't even have a global representation in closed form and need to be described in a piecewise manner using only one of these forms. The classical schemes for that purpose are boundary decomposition with surface patches and constructive point-set algebras of implicit primitives.

We propose a new framework for geometric modeling that integrates the parametric / implicit representations and can be used to describe complex shapes. This framework is based on adaptive simplicial decompositions of the embedding space and conforming piecewise diffeomorphisms.

The research reported here is part of the Ph.D. thesis work of Vinicius Mello under the supervision of Luiz Velho.

2. Iso-Simplicial Model

Our framework starts with the notion of Iso-Simplicial Model (ISM).

Definition 1 An *Iso-Simplicial Model*, O consists of:

1. A polyhedron $\mathcal{P} \subset R^n$ triangulated by the simplicial complex K;
2. For each n-simplex $\sigma \in K$, a simplicial diffeomorphism $X_\sigma : \sigma \to \Delta_n$.
3. A discrete function f from the vertices of K to the set $R - \{0\}$.

where Δ_n is the *standard n-simplex* defined by

$$\Delta_n = \{(w_0, \ldots, w_n) \in \Pi^n | w_i \geq 0\}.$$

with $\Pi^n = \{(w_0, \ldots, w_n) \in R^{n+1} | w_0 + \cdots + w_n = 1\}$.

Furthermore, for each n-simplex $\sigma =< p_0, \ldots p_n >$ in the Euclidean space $\mathbb{R}^n$, we can associate a map $W_\sigma : \Pi \to \mathbb{R}^n$ given by

$$W_\sigma(w_0, \ldots, w_n) = w_0 p_0 + \ldots + w_n p_n$$

that is called *barycentric parametrization*. This map has an inverse W_σ^{-1} known as *barycentric coordinates* with respect to σ. Note that W_σ takes Δ_n in σ.

Item 1 provides a piecewise simplicial structure for the model. Item 2 is essentially a reparametrization of each n-simplex compatible with the neighborhood relations in K. Item 3 gives a classification function for the vertices of K relative to the object.

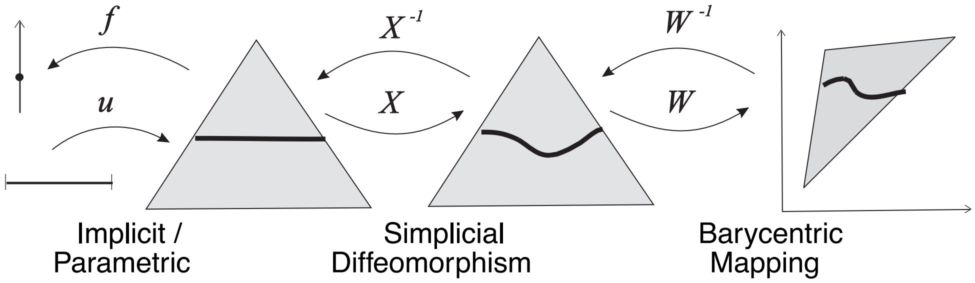

Figure 1: *Parametric-Implicit representation using simplicial diffeomorphisms.*

From an ISM, we can construct both an implicit and parametric representation that describes a solid object. In particular, for each n-simplex, σ, we can define the implicit function

$$f_\sigma(p) = \overline{f} \circ X_\sigma^{-1} \circ W^{-1}(p) \quad (1)$$

and the parametric function

$$g_\sigma(u) = W \circ X_\sigma(u) \quad (2)$$

where $\overline{f}$ is the piecewise linear extension of f, $p \in \mathbb{R}^n$, and $u \in U_\sigma$, is the iso-simplicial barycentric coordinate associated with σ. (See Figure 1.)

Thus, the implicit representation is $F(p) = \sum_{\sigma \in K} f_\sigma(p)$, and the parametric representation is $G(u) = \cup g_\sigma(u)$, with $u \in U_\sigma$.

The equation $F(p) = 0$ defines a hypersurface which topology can be recovered from the topology of K and from the signal of function f, and which its geometry depends on the diffeomorphisms X_σ. The parametric domain $\cup U_\sigma$ is given by the piecewise linear 0-isocontour of the function $\overline{f}$.

3. Bernstein-Bézier Simplicial Diffeomorphisms

The key issue related to Implicit Simplicial Models is how to define a map X which is guaranteed to be a diffeomorphism and preserves the structure of the simplicial complex K. More formally, X has to be a *K-invariant diffeomorphism.*

Definition 2 Let K be a simplicial complex, and $X\colon |K| \to |K|$ a differentiable function. We say that X is a simplicial diffeomorphism with respect to the complex K, or simply a *K-invariant diffeomorphism*, if, for each simplex $\sigma \in K$,

1. $X(\sigma) \subset \sigma$.
2. $D(X|_{\mathrm{int}(\sigma)})(p)$ is injective for all $p \in \mathrm{int}(\sigma)$

Denoting by $\mathcal{D}_\mathcal{S}(K)$ the set of all K-invariant diffeomorphisms, it is clear that:

$$X \in \mathcal{D}_\mathcal{S}(K) \Rightarrow X^{-1} \in \mathcal{D}_\mathcal{S}(K);$$
$$X_1, X_2 \in \mathcal{D}_\mathcal{S}(K) \Rightarrow X_1 X_2 \in \mathcal{D}_\mathcal{S}(K).$$

The main problem now is to construct a simplicial diffeomorphism which has a suitable computational form. In order to solve this problem we have chosen to work with homogeneous polynomial mappings. $H := (H^1, \dots H^n) : \mathbb{R}^n \to \mathbb{R}^n$, in the basis $H = \sum_{|I|=m} b_I B_I$, where b_I are the control points of H, and $B_I := \binom{m}{I} W^I$ are the Bernstein-Bézier polynomials of degree m.

Our goal is to find simple restrictions on the control points b_I, such that H is a K-invariant diffeomorphism. Indeed, we have shown [Mel05] that the desired sufficient conditions are:

- All b_I need to be contained in their associated faces of Δ_n;
- For each face of Δ_n, only the central b_I is allowed to move.

Note that, since composition of diffeomorphisms is also a diffeomorphism, the above result is constructive and general. Figure 2 shows an example of a surface modeled with simplicial diffeomorphism.

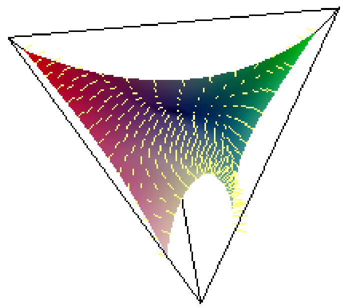

Figure 2: *Surface modeled with simplicial diffeomorphism.*

References

[Mel05] Mello V.: *Novos Metodos Simpliciais em CG.* PhD thesis, IMPA - (in preparation), 2005.