Appendix A. Derivations and Proofs

A.1. Proof of Lemma 1 (Nature of Min. Energy Path)

Proof: Consider a small segment of length dx along the path. Assuming a speed of $\mathbf{v_x}$ along that segment, the total energy expended to traverse the distance dx is equal to: $E_x = m\int(e_s + e_w|\mathbf{v_x}|^2)dt = m(e_s + e_w|\mathbf{v_x}|^2)(dx/|\mathbf{v_x}|) = m(e_s/|\mathbf{v_x}| + e_w|\mathbf{v_x}|)dx$. In order to minimize the energy, $\frac{\partial E_x}{\partial |\mathbf{v_x}|}$ = 0 implies $m(-e_s/|\mathbf{v_x}|^2 + e_w)dx = 0$. Therefore, $|\mathbf{v_x}| = \sqrt{(e_s/e_w)}$. In order to minimize the total energy expended, the agent needs to traverse each segment of length dx (and hence the whole path) at a speed of $\sqrt{(e_s/e_w)}$. For a total path length of L, the total energy expended evaluates to $m(\sqrt{(e_s e_w)} + \sqrt{(e_s e_w)})L = 2mL\sqrt{(e_s e_w)}$. The above statement also implies that the agent needs to take the shortest path available from its source to destination in order to reduce the total distance traversed, and correspondingly the total effort (or energy) expended.

A.2. Objective Function of Eqn. 4 is a convex function

Refer to Section 3.1 for notations and figures. $E(\mathbf{v}_A^{new}) = m\tau(e_s + e_w|\mathbf{v}_A^{new}|^2) + 2m|\mathbf{G}_A - \mathbf{p}_A - \tau\mathbf{v}_A^{new}|\sqrt{(e_s e_w)}$
$= m\tau e_s + m\tau e_w|\mathbf{v}_A^{new}|^2 + 2m\tau\sqrt{(e_s e_w)}|\mathbf{v}_A^{new} - (\mathbf{G}_A - \mathbf{p}_A)/\tau|$
It follows from first principles of convex functions [BV04] that $|\mathbf{v}_A^{new}|^2$ and $|\mathbf{v}_A^{new} - (\mathbf{G}_A - \mathbf{p}_A)/\tau|$ are individually convex functions. A weighted sum (with all positive weights) of convex functions is also a **convex function**. Since both $m\tau e_w$ and $m\tau\sqrt{(e_s e_w)}$ are greater than *zero*, $E(\mathbf{v}_A^{new})$ is convex.

A.3. Minima of Eqn. 4 lies on the boundary of PV_A

It follows from Lemma 1 that $\mathbf{v}_A^{des} = \sqrt{(e_s/e_w)}(\widehat{\mathbf{G}_A - \mathbf{p}_A})$. Let $\mathbf{v}_A^{new} = (x,y)$. To find the minima of the objective function, we set $\frac{\partial E(\mathbf{v}_A^{new})}{\partial x} = 0$ and $\frac{\partial E(\mathbf{v}_A^{new})}{\partial y} = 0$, which implies $x/y = (\mathbf{G}_A - \mathbf{p}_A)_x/(\mathbf{G}_A - \mathbf{p}_A)_y$. Also, $x^2 + y^2 = e_s/e_w$. Hence, $\mathbf{v}_A^{new} = \mathbf{v}_A^{des}$. In case $\mathbf{v}_A^{des} \notin PV_A$, we need to compute the optimal point within the region of permissible velocities (PV_A). We now prove $\mathbf{v}_A^{new}$ lies on a linear boundary segment by contradiction. Assume the optimal velocity $\mathbf{v}'$ (= $\mathbf{v}_A^{new}$) lies strictly inside the PV_A region. Consider the segment joining $\mathbf{v}'$ to $\mathbf{v}_A^{des}$. Since $E(\mathbf{v}_A^{new})$ is convex, its projection function along any line would also be convex [BV04]. Since $\mathbf{v}_A^{des}$ is the *global minimum*, $E(\mathbf{v}_A^{new})$ is *strictly increasing* along the line segment from $\mathbf{v}_A^{des}$ to $\mathbf{v}'$. Since $\mathbf{v}'$ is inside PV_A, the segment intersects the PV_A at a point for which the objective function evaluates to a smaller value. Hence $\mathbf{v}'$ is not the optimal value, and we have arrived at a contradiction.

A.4. Proofs of Lemma 2 and Lemma 3

Proof of smoothness (**Lemma 2**). **Proof:** To show that the trajectories generated are C1-continuous, we need to prove that the paths are C0-continuous, and their derivative (i.e. velocity) is also C0-continuous. We first assume that discrete time steps of the underlying simulation approach zero in the limit. Our simulation displaces the agent by the product of the instantaneous agent velocity and the time change (Euler integration). Since time varies C0-continuously, the agent traverses C0-continuous trajectory. In order to prove that the velocity of the agent is C0-continuous, we need to prove that our energy minimization formulation (Eqn. 4) computes velocities that vary in a C0-continuous fashion.

Consider the agent A. We assume that the region of permissible velocities changes in a C0-fashion (i.e. for any boundary curve of PV_A, and a point on that curve, the path traced out by that point, with change in time, is C0-continuous). Furthermore, the boundary curves themselves are continuous, with at least C0 continuity at their end points

Consider the boundary segment along which the energy function is minimized. Note that all the coefficients in Eqn. 4 are either constant or vary with the positions of the agent and its neighbors. To minimize Eqn. 4, we set the partial derivative of the objective function of Eqn. 4 to be equal to zero. This results in finding the roots of a polynomial equation, whose coefficients trace a C0-continuous path. A polynomial equation with C0-continuous coefficients also has C0-continuous roots [Coo08]. Hence as long as the minimum lies on a specific boundary curve, the path traced by the velocity is C0-continuous. Furthermore, as the minima changes from one boundary curve to another curve, the partial derivative at their common end point should also evaluate to zero (follows from the C0-continuity of the PV_A boundary curves at their end points).

Lemma 3: The total effort expended by an agent using our optimization formulation is *guaranteed* to be within $(\pi/2)/(1-\rho)$ of its *optimal* total expended effort.
Proof: Assume there exists a straight line path from the source to the destination of length L. Furthermore, we assume that the start and the goal positions of the agent are not congested. During the course of the simulation, the agent moves through phases of non-congestion (at speed $\sqrt{(e_s/e_w)}$) and congestion (for ρ of the simulation time). During non-congestion, the agent makes progress by expending the minimum amount of energy towards its goal. During congestion, the underlying collision avoidance algorithm provides a set of velocities to make progress towards the goal. Although the velocity may not be directed towards the goal, we assume that the system assures forward progress. In the worst case, the agent may move in a direction tangential to the desired one – thereby traversing a semicircle. Hence, the total distance traversed by the agent maybe $(\pi/2)$ times greater than the shortest possible distance. Since the agent is in congestion for a fraction ρ of total simulation time, the total amount of expended energy is less than $2L\sqrt{(e_s e_w)})(\pi/2)/(1-\rho)$, which is not more than $(\pi/2)/(1-\rho)$ times the least possible energy possible.

References

[BV04] Boyd S., Vandenberghe L.: *Convex Optimization*. Cambridge University Press, 2004.

[Coo08] Coolidge J. L.: The continuity of the roots of an algebraic equation. In *The Annals of Mathematics* (1908), vol. 9, pp. 116–118.