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Design of dynamical stability properties in character animation

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Abstract

Human movements and the collective behavior of interacting characters in crowds can be described by nonlinear dynamical systems. The design of stability properties of dynamical systems has been a core topic in control theory and robotics, but has rarely been addressed in the context of computer animation. One potential reason is the enormous complexity of the dynamical systems that are required for the accurate modeling of human body movements, and even more for the interaction between multiple interacting agents. We present an approach for the online simulation of realistic coordinated human movements that exploits dynamical systems that are simple enough in order to permit a systematic treatment of their stability. We introduce contraction theory as a novel framework that permits a systematic treatment of stability problems for systems in character animation. It yields tractable global stability conditions, even for systems that consist of many nonlinear interacting modules or characters. We show some first simple applications of this framework for the animation of coordinated behavior in groups of interacting human characters.

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1. Introduction

The online simulation of human behavior is a core problem in computer animation with important applications such as computer games. While the dominant approach for the generation of realistic human movements is based on offline synthesis using motion capture, this approach cannot easily be transferred to real-time applications. Only recently, researches have started to develop methods in order to learn models for online synthesis from motion capture data [SHP04, SLSG01, HPP05]. Dynamical systems derived, for example, from biomechanical or physical models seem particularly appropriate for real-time synthesis [TW90, GTH98, BRI06]. However, it has turned out that such models for the generation of human movements with high degree of realism typically have to be rather detailed [Ter09, HWBO95, AHS03], resulting in complex dynamical systems whose properties are difficult to control. Consequently, the dynamical stability properties of such systems have rarely been addressed, and given their complexity it is an open question whether they can be treated at all.

An important domain of the application of dynamical systems in computer animation is the simulation of autonomous and collective behavior of many characters, e.g. in crowd animation [MT01, TCP06]. Some work in this domain has been inspired by observations in biology showing that coordinated behavior of large groups of agents, such as flocks of birds, can be modeled at an emergent behavior that results from the dynamic interactions between individual agents, without requiring a central mechanism that ensures coordination [Cou09, CDF*01]. One example is the tendency of multiple agents to synchronize their behavior, for example during walking or applauding. It is well-known that such behaviors can be analyzed efficiently within the framework of nonlinear dynamics [PRK03]. This makes it in-

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teresting to exploit the underlying principles for the automatic synthesis of collective behavior in computer animation [TWP03, Rey99, BH97]. So far, the design of such technical systems has been often heuristic, exploiting empirical results from simulations instead of deriving the system parameters from theoretical results on the system dynamics. However, the controlled engineering of such system makes more systematic theoretically founded approaches highly desirable. A critical limiting factor in this context is the complexity of the dynamical models for individual agents or characters, which often makes a systematic treatment of stability properties infeasible. In previous work, we have developed a method that approximates complex human behavior by relatively simple nonlinear dynamical systems. Consistent with related approaches in robotics [GRIL08, AMS97, SIB03] and biology [FH05], this method generates complex movements by combination learned movement primitives [PMOG08]. The resulting system architecture is rather simple and thus suitable for a treatment of dynamical stability properties. While the design of stability properties is a central topic in robotics [BRI06, RI06, GRIL08], it is rarely addressed in character animation.

The goal of this article is to introduce a novel framework for the analysis and design of the stability properties of systems for interactive character animation. Exploiting models that are based on learned movement primitives, we obtain a system dynamics that can be analyzed, even for situations with multiple characters.

We introduce *contraction theory* as new theoretical approach that permits a treatment of the dynamical properties of networks of coupled nonlinear dynamical elements. Contraction theory has been applied successfully before to analyze other types of complex systems [LS98, Slo03, WS05, PS07].

The paper is structured as follows: After a brief discussion of related approaches (Section 2), we first introduce some basic concepts from contraction theory (Section 3) and derive more detailed mathematical results that are important for our applications (Section 4). We then discuss several applications of these results in the context of our learning-based real-time animation system (Section 5), followed by some conclusions.

2. Related Work

The dynamics of collective behaviors of animals has been analyzed extensively in biology, e.g. for collective motions in flocks [Cou09, CDF*01]. These observations have inspired a variety of approaches in particular in robotics, where group coordination and cooperative control have been studied in the context of the navigation of groups of vehicles, or the control of agents that realize coordinated behavior for useful tasks (e.g. [Mat95, BH97]). The dynamics of interactive group behavior has been rarely discussed in the field of computer animation [Rey99]. However, the simulation of collective behavior by self-organization in systems of dynamically coupled agents seems interesting for several reasons: First, it might help to reduce the computational costs of traditional computer animation techniques, such as scripting or path planning [TCP06, TWP03]. In addition, the generation of collective behavior by self-organization results in spontaneous adaptation to perturbations or changes in the number of characters [CS07, OEH02].

3. Basic mathematical methods

Our animation systems models the behavior of characters by the learning of mappings between the stable solutions of relatively simple nonlinear dynamical systems and the complex trajectories of real human behaviors [GMP*09]. More detailed physical or biomechanical models of such complex human movements are typically rather complex (requiring around 30 degrees of freedom for sufficient realism [HWBO95, GTH98]). Nonlinear systems of this complexity typically are not accessible for a more detailed analysis of their dynamical stability properties. By learning of appropriate simplified models our approach results in dynamical models with a limited degree of complexity. Using appropriate mathematical methods, this makes it possible to develop a framework for the analysis and design of their stability properties.

In the following a single character will be described by a dynamical system of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \tag{1}$$

where the variable $\mathbf{x}$ signifies the dynamical state of the character. For a walking avatar, this dynamics could be given by a limit cycle oscillator, whose periodic solution is mapped onto the joint angles of the character. The nonlinear mapping between dynamical state $\mathbf{x}$ and the joint angles is learned using kernel methods (see Section 5.1 for details). The learned nonlinear mapping is bounded and acts as nonlinear observer of the state variable that does not modify the overall stability properties of the system, unless the joint angles are fed back into the dynamical state. The dynamics (1) can also be interpreted as describing a *central pattern generator* that drives the movement of the character.

In the following we will describe methods from *contraction theory* [LS98] that are useful for the analysis of the stability of networks of such central pattern generators. Specifically, we will introduce the basic definitions of the contraction theory and will rehearse some major theorems that are central for derivation of new results in Section 4.

3.1. Notations

In the following we will derive bounds for the convergence of solutions of the dynamical system of the form (1). These bounds depend of the eigenvalues of matrices that are derived from the *Jacobian* of the system $\mathbf{J}(\mathbf{x},t) = \frac{\partial \mathbf{f}(\mathbf{x},t)}{\partial \mathbf{x}}$. Given a square matrix $\mathbf{A}$, in the following the matrix $\mathbf{A}_s = (\mathbf{A} + \mathbf{A}^T)/2$ signifies its symmetric part. Moreover, we define the real-valued matrix functions $\lambda_{\min}(\mathbf{A}_s)$ and $\lambda_{\max}(\mathbf{A}_s)$ which correspond to the smallest, respectively largest, eigenvalue of the symmetric matrix $\mathbf{A}_s$. This implies that the matrix $\mathbf{A}_s$ is positive definite (denoted by $\mathbf{A}_s > \mathbf{0}$) if $\lambda_{\min}(\mathbf{A}_s) > 0$, and negative definite (denoted by $\mathbf{A}_s < \mathbf{0}$) if $\lambda_{\max}(\mathbf{A}_s) < 0$. The matrix inequality $\mathbf{A} > \mathbf{B}$ denotes $\lambda_{\min}(\mathbf{A}_s) > \lambda_{\max}(\mathbf{B}_s)$.

If the matrix is itself a function of state and time (i.e., $\mathbf{A}_s(\mathbf{x},t)$) we say that it is *uniformly positive definite* if there exists a real $\beta > 0$ such that $\forall \mathbf{x}, \forall t : \lambda_{\min}(\mathbf{A}_s(\mathbf{x},t)) \geq \beta$. Likewise, we say it is *uniformly negative definite* if there exists a $\beta > 0$ such that $\forall \mathbf{x}, \forall t : \lambda_{\max}(\mathbf{A}_s(\mathbf{x},t)) \leq -\beta$.

3.2. Contracting dynamical systems

Dynamical systems describing the behavior of autonomous characters are essentially nonlinear. This makes the analysis of their stability properties for systems including multiple interacting characters a very difficult problem. A major difficulty of this analysis is that for nonlinear, as opposed to linear dynamical systems, stability properties of parts usually do not transfer to composite systems. Contraction theory [LS98] provides a general method for the analysis of essentially nonlinear systems, which permits such a transfer, making it suitable for the analysis of complex systems with many components. The classical approach for the stability analysis of nonlinear systems is to compute first the stationary solutions of the dynamics, and then to establish its *local stability* by linearization in the neighborhood of this solution. Already the computation of stationary solutions is often difficult or possible only numerically. Contraction theory takes a different approach and characterizes the system stability by the behavior of the differences between solutions with different initial conditions. If these differences vanish exponentially over time, and its solution converges towards a single trajectory, independent from the initial states, the system is called *globally asymptotically stable*. Interestingly, the analysis of such differences between solutions is often simpler than the classical linearization approach, making systems tractable that would be impossible to analyze with the classical approach.

More concretely, assume $\mathbf{x}(t)$ is one solution of the system and $\tilde{\mathbf{x}}(t) = \mathbf{x}(t) + \delta\mathbf{x}(t)$ a neighboring one. The function $\delta\mathbf{x}(t)$ is also called *virtual displacement* (see Fig.1). If the virtual displacement is small enough the last equation together with equation (1) implies

$$\delta\dot{\mathbf{x}}(t) = \mathbf{J}(\mathbf{x},t)\delta\mathbf{x}(t)$$

implying through $\frac{d}{dt}||\delta\mathbf{x}(t)||^2 = 2\delta\mathbf{x}^T(t)\mathbf{J}_s(\mathbf{x},t)\delta\mathbf{x}$ the inequality:

$$||\delta\mathbf{x}(t)|| \leq ||\delta\mathbf{x}(0)||\, e^{\int_0^t \lambda_{\max}(\mathbf{J}_s(\mathbf{x},s))\, \mathrm{d}s} \tag{2}$$

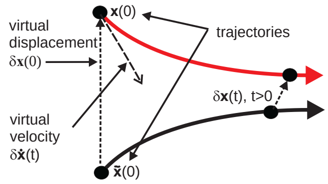

Figure 1: *Two trajectories of a dynamical system and the virtual displacement.*

If the Jacobian is uniformly negative definite this equation implies that any nonzero virtual displacement decays exponentially to zero over time. This decay occurs with a *convergence rate* (inverse timescale) that is bounded from below by the quantity $\rho_c = -\sup_{\mathbf{x},t} \lambda_{\max}(\mathbf{J}_s(\mathbf{x},t))$. By 'concatenating' such virtual displacements at fixed points in time one can show that any difference between the trajectories decays to zero with at least this time constant [LS98]. This has the consequence of all trajectories converging towards a single trajectory exponentially. Therefore, this motivates:

Definition 1 With respect to the dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},t)$, the regions in state space for which the symmetrized Jacobian $\mathbf{J}_s = \frac{1}{2}(\frac{\partial \mathbf{f}}{\partial \mathbf{x}} + \frac{\partial \mathbf{f}}{\partial \mathbf{x}}^T)$ is uniformly negative definite are called *contracting regions*. All solutions that start in these regions converge towards a single trajectory for $t \to \infty$.

The previous argumentation can be extended by measuring the length of the virtual displacement using a different metric (coordinate system). By assuming a uniformly invertible square matrix $\Theta(\mathbf{x},t)$, which in most cases is state- and time-dependent, one can introduce the transformed displacement $\delta\mathbf{z}(t) = \Theta(\mathbf{x},t)\delta\mathbf{x}(t)$. Analogous to the previous case one finds:

$$\frac{d}{dt}(\delta\mathbf{z}^T\delta\mathbf{z}) = 2\delta\mathbf{z}^T\delta\dot{\mathbf{z}} = 2\delta\mathbf{z}^T \underbrace{(\dot{\Theta} + \Theta\frac{\partial \mathbf{f}}{\partial \mathbf{x}})\Theta^{-1}}_{\mathbf{F}}\delta\mathbf{z}$$

This implies the following general result:

Theorem 1 Assume that for the system (1) it is possible to find a square matrix $\Theta(\mathbf{x},t)$ such that $\Theta(\mathbf{x},t)^T\Theta(\mathbf{x},t)$ is uniformly positive definite, and such that the *generalized Jacobian*

$$\mathbf{F} = (\dot{\Theta} + \Theta\frac{\partial \mathbf{f}}{\partial \mathbf{x}})\Theta^{-1} \tag{3}$$

is uniformly negative definite, then all system trajectories converge exponentially to a single trajectory, and the system is called *contracting*. The rate of convergence of $||\delta\mathbf{z}(t)||$ is at least $\rho_c = -\sup_{\mathbf{x},t}\lambda_{\max}(\mathbf{F}_s(\mathbf{x},t))$. The matrix $\mathbf{M}(\mathbf{x},t) = \Theta(\mathbf{x},t)^T\Theta(\mathbf{x},t)$ is also called the *contraction metric*.

Conversely, the existence of a uniformly positive definite metric $\mathbf{M}(\mathbf{x},t) = \Theta(\mathbf{x},t)^T\Theta(\mathbf{x},t)$ with respect to which the system is contracting is a necessary condition for the global exponential convergence of trajectories [LS98]. Furthermore, all transformations Θ corresponding to the same $\mathbf{M}$ result in the same eigenvalues for the symmetric part of $\mathbf{F}$ [Slo03], and thus the same convergence rate. (The proofs can be found in [LS98, Slo03].)

3.3. Partial contraction and flow-invariant manifolds

Many systems are not contracting with respect to all dimensions of the state space, but show convergence with respect to a subset of dimensions. A typical example is an externally driven nonlinear oscillator. By its tendency to self-initiate oscillatory solutions it is unstable, and thus non-contracting, within a region around the the origin of state space. However, independent of the initial state, it might converge exponentially against a single trajectory that is determined by the external driving signal. Partial contraction [WS05] allows to capture this property in a mathematically well-defined manner, and to derive from it results in the global stability of the system. The key idea is to construct an auxiliary system that is contracting with respect to a subset of dimensions (or submanifold) in state space.

Theorem 2 Consider a nonlinear system of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{x}, t) \tag{4}$$

and assume that the auxiliary system

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, \mathbf{x}, t) \tag{5}$$

is contracting with respect to $\mathbf{y}$ uniformly for all relevant $\mathbf{x}$. If a particular solution of the auxiliary system verifies a specific smooth property, then all trajectories of the original system (4) verify this property with exponential convergence. The original system is then said to be partially contracting.

A 'smooth property' is a property of the solution that depends smoothly on space and time, such as convergence against a particular solution or value. The proof of the theorem is immediate noticing that the observer-like system (5) has $\mathbf{y}(t) = \mathbf{x}(t)$ for all $t \geq 0$ as a particular solution. Since all trajectories of the $\mathbf{y}$-system converge exponentially to a single trajectory, this implies that also the trajectory $\mathbf{x}(t)$ verifies this specific property with exponential convergence.

Related to partial contraction are the following methods that will be crucial for the derivation of results on the synchronization of groups of avatars. Again starting form the equation (1), we assume the existence of a *flow-invariant linear subspace* $\mathcal{M}$, i.e. a linear subspace $\mathcal{M}$ such that $\forall t : \mathbf{f}(\mathcal{M}, t) \subset \mathcal{M}$). This implies that any trajectory starting in $\mathcal{M}$ remains in $\mathcal{M}$. Further, we assume that $p = \dim(\mathcal{M})$ and consider an orthonormal basis $(\mathbf{e}_1, ..., \mathbf{e}_n)$ where the first p vectors form a basis of $\mathcal{M}$ and the last $n-p$ a basis of $\mathcal{M}^\perp$, the orthogonal space of $\mathcal{M}$. We define an $(n-p) \times n$ matrix $\mathbf{V}$ whose rows are $\mathbf{e}_{p+1}^T, ..., \mathbf{e}_n^T$. This matrix can be regarded as projection on $\mathcal{M}^\perp$, which implies $\mathbf{x} \in \mathcal{M} \Leftrightarrow \mathbf{V}\mathbf{x} = 0$. It verifies $\mathbf{V}\mathbf{V}^T = \mathbf{I}_{n-p}$ and $\mathbf{V}^T\mathbf{V} + \mathbf{U}^T\mathbf{U} = \mathbf{I}_n$, where $\mathbf{U}$ is the matrix formed by the first p basis vectors.

Theorem 3 Assuming that for the dynamical system (1) a flow-invariant linear subspace $\mathcal{M}$ exists with the associated orthonormal projection matrix $\mathbf{V}$. A particular solution $\mathbf{x}_p(t)$ of this system converges exponentially to $\mathcal{M}$ if the auxiliary system

$$\dot{\mathbf{y}} = \mathbf{V}\mathbf{f}\left(\mathbf{V}^T\mathbf{y} + \mathbf{U}^T\mathbf{U}\mathbf{x}_p(t), t\right) \tag{6}$$

is contracting with respect to $\mathbf{y}$ for all relevant $\mathbf{x}_p$, then starting from any initial conditions, all trajectories of the original system will exponentially converge to the invariant subspace $\mathcal{M}$. If furthermore all the contraction rates for (6) are lower-bounded by some constant $\lambda > 0$, uniformly in $\mathbf{x}_p$ and in a common metric, then the convergence to $\mathcal{M}$ will be exponential with a minimum rate λ.

The proof for this theorem can be found in [PS07]. It implies that a simple sufficient condition for global exponential convergence to $\mathcal{M}$ is given by the following inequality that needs to hold uniformly:

$$\mathbf{V}\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)_s \mathbf{V}^T < \mathbf{0} \tag{7}$$

An even more general condition can be derived if there exists a constant invertible transform Θ on $\mathcal{M}^\perp$ such that

$$\Theta\mathbf{V}\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)_s \mathbf{V}^T\Theta^{-1} < \mathbf{0} \tag{8}$$

is fulfilled uniformly [PS07].

3.4. Linear coupling of dynamical primitives

The techniques described previously can be applied to analyze the stability of networks of coupled dynamical elements, such as oscillators. We applied this approach in order to analyze the stability of groups of characters that are coupled in order to coordinate their behaviors.

We assume in the following n systems with linear, so called *diffusive coupling* in the form (cf. [WS05]):

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i, t) + \sum_{j \neq i} \mathbf{K}_{ij}(\mathbf{x}_j - \mathbf{x}_i) \quad \forall i = 1, ..., n \tag{9}$$

In the case of diffusive coupling, after synchronisation, the dynamics of each subsystem is equivalent to the dynamics of uncoupled subsystem.

The matrix $\mathbf{L}$ with the blocks ($\mathbf{L}_{ii} = \sum_{j \neq i} \mathbf{K}_{ij}$ and $\mathbf{L}_{ij} = -\mathbf{K}_{ij}$ for $j \neq i$) is called *Laplacian matrix* of the coupling graph. With this matrix and the definitions $\mathbf{x} = [\mathbf{x}_1^T, ..., \mathbf{x}_n^T]^T$ and $\mathbf{f}(\mathbf{x}_i, t) = [\mathbf{f}(\mathbf{x}_1, t)^T, ..., \mathbf{f}(\mathbf{x}_n, t)^T]^T$ the equation system can be written in vector form: $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) - \mathbf{L}\mathbf{x}$. This implies that the Jacobian of the system is given by $\mathbf{J}(\mathbf{x},t) = \mathbf{D}(\mathbf{x},t) - \mathbf{L}$, where

$$\mathbf{D}(\mathbf{x},t) = \begin{bmatrix} \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}_1,t) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}_n,t) \end{bmatrix}. \quad (10)$$

For the case of diffusive coupling, we assume again the existence of a flow-invariant linear subspace $\mathcal{M}$ of the $\mathbf{x}$ space that contains a particular solution of the form $\mathbf{x}_1^* = \cdots = \mathbf{x}_n^*$. For this solution all state variables $\mathbf{x}_i$ are identical and thus in synchrony. In addition, for this solution the coupling term in equation (9) vanishes so that the form of the solution is identical of the solution of the uncoupled systems $\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i,t)$.

From the last secition, it can be implied that $\mathbf{V}$ is a projection matrix onto the subspace $\mathcal{M}^{\perp}$, a sufficient condition for convergence toward this solution, which is the matrix inequality $\mathbf{V}(\mathbf{D}(\mathbf{x},t) - \mathbf{L})_s \mathbf{V}^T < \mathbf{0}$. From this inequality the following sufficient condition for exponential convergence can be derived [PS07]

$$\lambda_{\min}(\mathbf{V}\mathbf{L}_s\mathbf{V}^T) > \sup_{\mathbf{x},t} \lambda_{\max}\left(\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x},t)\right)_s\right) \quad (11)$$

which implies the following minimum convergence rate: $\rho_c = -\sup_{\mathbf{x},t} \lambda_{\max}(\mathbf{V}(\mathbf{D}(\mathbf{x},t) - \mathbf{L})_s \mathbf{V}^T)$.

4. Mathematical results

The animation systems discussed in the following exploit character models whose behavior is driven by nonlinear limit cycle oscillators. The stationary solution of these oscillators is given by a sinusoidal oscillation with a constant equilibrium amplitude. Groups of interacting characters can be modeled by coupled networks of such nonlinear oscillators. In the following, we describe how methods from contraction theory can be exploited to analyze the dynamics of such networks, providing mathematical results that help to design the behavior of animations of the collective behavior of such interacting characters.

4.1. Andronov-Hopf oscillator

The dynamics of an individual character is modeled by an Andronov-Hopf oscillator, a nonlinear oscillator whose choice of parameters, is characterized by a limit cycle that corresponds to a circular trajectory in phase space. For appropriate re-parametrization (rescaling of time and state-space axes) the dynamics of this oscillator is described by the differential equations [AVK87]:

$$\begin{cases} \dot{x}(t) = \left(1 - \left(x^2(t) + y^2(t)\right)\right) x(t) - \omega y(t) \\ \dot{y}(t) = \left(1 - \left(x^2(t) + y^2(t)\right)\right) y(t) + \omega x(t) \end{cases} \quad (12)$$

which can be written compactly in vector form (with $\mathbf{x} = [x,y]^T$):

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x},t) \quad (13)$$

Introducing polar coordinates $r(t) = \sqrt{x^2(t) + y^2(t)}$ and $\phi(t) = \arctan(y(t)/x(t))$, this system can be rewritten:

$$\begin{cases} \dot{r}(t) = & r(t)\left(1 - r^2(t)\right) \\ \dot{\phi}(t) = & \omega \end{cases}$$

The symmetrized Jacobian of this system is given by

$$\mathbf{J}_s = \begin{bmatrix} 1 - 3r^2 & 0 \\ 0 & 0 \end{bmatrix}$$

showing that, according to Definition 1, this system is semi-contracting [PS07] in the region $|r| > 1/\sqrt{3}$ where its symmetrized Jacobian is uniformly negative definite. A more general result can be obtained by using a different metric (cf. Section 4.2). Introduction of the new variable $\rho = 1/r^2 > 0$ transforms the dynamics into the form:

$$\dot{(r^2)} = 2r^2\left(1 - r^2\right) \Rightarrow \dot{\rho} = 2(1 - \rho)$$

In this case, one of the eigenvalues of the symmetrized Jacobian are -1 and 0, so that the system is semi-contracting in the whole phase plane, excluding the points with $\rho = 0$.

4.2. Symmetric coupling of two oscillators

The constraints that guarantee the synchronization of two symmetrically coupled oscillators can be proven following [PS07]. The dynamics of two Hopf oscillators with symmetric diffusive linear coupling is given (using $\mathbf{x}_i = [x_i, y_i]^T$, $i = 1,2$, and the definition according to equation (13)) by:

$$\begin{pmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{f}(\mathbf{x}_1) \\ \mathbf{f}(\mathbf{x}_2) \end{pmatrix} - k \underbrace{\begin{bmatrix} \mathbf{I} & -\mathbf{I} \\ -\mathbf{I} & \mathbf{I} \end{bmatrix}}_{\mathbf{L}_{(2)}} \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix}$$

$$\Leftrightarrow \quad \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) - k\,\mathbf{L}_{(2)}\mathbf{x} \quad (14)$$

According to the definition discussed in Section 3.3, a flow-invariant manifold $\mathcal{M}$ of this system is given by the linear 2-dimensional subspace that is defined by the linear relationship $\mathbf{x}_1 = \mathbf{x}_2$. For points on this manifold the coupling term vanishes, and the solution of the coupled system coincides with the solutions of the uncoupled individual oscillators.

By interchanging the columns of the matrix $\mathbf{L}_{(2)} - \lambda\mathbf{I}$ it is easy to show that $\det(\mathbf{L}_{(2)} - \lambda\mathbf{I}) = \lambda^2(\lambda - 2)^2$. This implies that the matrix $\mathbf{L}_{(2)}$ has rank 2. Its nullspace is 2-dimensional and thus coincides with $\mathcal{M}$. If according to Section 3.3 the matrix $\mathbf{V}$ is a projector onto $\mathcal{M}^{\perp}$ this implies that the matrix $\mathbf{V}\mathbf{L}_{(2)}\mathbf{V}^T$ has only the eigenvalues 2.

The Jacobian for a single oscillator is given by

$$\mathbf{J}(\mathbf{x}_i) = \frac{\partial \mathbf{f}}{\partial \mathbf{x}_i} =$$

$$\begin{bmatrix} -(x_i^2 + y_i^2 - 1) - 2x_i^2 & -2x_iy_i - \omega \\ -2x_iy_i + \omega & -(x_i^2 + y_i^2 - 1) - 2y_i^2 \end{bmatrix}$$

implying $|\mathbf{J}_s(\mathbf{x}_i) - \lambda\mathbf{I}| = (1 - r^2 - \lambda)(1 - 3r^2 - \lambda)$ with

$r^2 = x_i^2 + y_i^2$. The eigenvalues of the matrix $\mathbf{J}_s(\mathbf{x}_i)$ are thus bounded by 1 from above.

Using the derived bounds for the eigenvalues, a sufficient condition for global exponential convergence of the coupled oscillator system can be derived from equation (11):

$$\lambda_{\min}\left(\mathbf{V}(k\mathbf{L}_{(2)})\mathbf{V}^T\right) = 2k > \sup_{\mathbf{x},t}\lambda_{\max}\left(\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)_s\right) = 1 \quad (15)$$

This implies that a sufficiently strong coupling with $k > 1/2$ guarantees the global exponential convergence against a stable behavior.

4.3. Symmetric all-to-all coupling of N oscillators

The last analysis can be extended for to any number N of coupled oscillators. In this case, the $2N$-dimensional square matrix $\mathbf{L}$ has the form:

$$\mathbf{L} = \begin{bmatrix} (N-1) & 0 & -1 & 0 & \dots \\ 0 & (N-1) & 0 & -1 & \dots \\ -1 & 0 & (N-1) & 0 & \dots \\ 0 & -1 & 0 & (N-1) & \dots \\ \dots & & & & \end{bmatrix}$$

i.e., $L_{ii} = N-1$ and $L_{ij} = -1$ if $i \neq j$ and $(i+j)\text{mod}_2 = 0$, and $L_{ij} = 0$ otherwise. By rearranging of the columns and rows this matrix can be restructured in the form:

$$\mathbf{L} = \begin{bmatrix} \mathbf{L}_G & 0 \\ 0 & \mathbf{L}_G \end{bmatrix} \quad (16)$$

where:

$$\mathbf{L}_G = \begin{bmatrix} (N-1) & -1 & -1 & \dots \\ -1 & (N-1) & -1 & \dots \\ -1 & -1 & (N-1) & \dots \\ \dots & & & \end{bmatrix} \quad (17)$$

Note that $\mathbf{L}_G = N\mathbf{I} - \mathbf{1}\mathbf{1}^T$. The matrix $\mathbf{1}\mathbf{1}^T$ has rank 1 and the eigenvector $\mathbf{1}$ with the eigenvalue N, while all other eigenvalues are 0. From $\det(\mathbf{L}_G - \lambda\mathbf{I}) = \det(-\mathbf{1}\mathbf{1}^T - (\lambda - N)\mathbf{I}) = 0$ follows that the matrix $\mathbf{L}_G$ has one eigenvalue 0 and all other $N-1$ eigenvalues are N. From this follows with equation (16) that two eigenvalues of the matrix $\mathbf{L}$ are 0, while all non-zero eigenvalues are N.

As for equation (15) one obtains the inequality $Nk > \sup_{\mathbf{x},t}\lambda_{\max}\left(\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)_s\right) = 1$. Global exponential convergence to a stable synchronized solution is thus guaranteed for $k > 1/N$.

4.4. Symmetric couplings with more general structure

Following the procedure in [WS05], we discuss in the following systems with more general symmetric couplings of N oscillators, where we assume equal coupling gains k. The corresponding dynamics is:

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i) + k\sum_{j\in\mathcal{N}_i}\left(\mathbf{x}_j - \mathbf{x}_i\right), \quad \forall i = 1,\dots,N \quad (18)$$

where $\mathcal{N}_i$ denotes the set of indices of all oscillators that are coupled with oscillator i. The couplings are assumed to be bidirectional, defining an undirected graph of couplings. This implies $j \in \mathcal{N}_i$ iff $i \in \mathcal{N}_j$. By construction the coupling graph is *balanced*, i.e. the sum of the (weighted) connections towards each oscillator equals the sum of (weighted) connections away from this oscillator. The corresponding symmetric Laplacian matrix $\mathbf{L}$ a block structure, where the blocks at positions (i,i) are given by $-\mathbf{I}$ and where the i-th diagonal block is given by $n_i\mathbf{I}$, n_i signifying the number of elements in $\mathcal{N}_i$.

Like in the previous sections, this matrix, by appropriate sorting of columns and rows, can be brought in the form $\begin{bmatrix} \mathbf{L}_G & 0 \\ 0 & \mathbf{L}_G \end{bmatrix}$, where $\mathbf{L}_G$ is called Laplacian matrix of the coupling graph. Since the network is balanced, the sum of the rows of this matrix are zero. This implies that $\mathbf{1}$ is an eigenvector with eigenvalue 0. Again, the block structure implies that all eigenvalues of $\mathbf{L}_G$ appear twofold in the matrix $\mathbf{L}$. Consequently, two of its eigenvalues are zero, independently of the form of the sets $\mathcal{N}_i$.

Following again the argumentation in the last sections one can derive a necessary condition for the exponential convergence from equation (11):

$$\lambda_{\min}\left(\mathbf{V}(k\mathbf{L})_s\mathbf{V}^T\right) = \lambda_{\mathbf{L}}^+ > \sup_{\mathbf{x},t}\lambda_{\max}\left(\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)_s\right) = 1 \quad (19)$$

Here, $\lambda_{\mathbf{L}}^+$ signifies the smallest non-zero eigenvalue of the matrix $\mathbf{L}_G$ that depends on the form of the coupling. The matrix $\mathbf{V}$ defines the projection to orthogonal complement of the flow invariant manifold $\mathbf{x}_1 = \dots = \mathbf{x}_n$. The condition for exponential convergence is thus $k > 1/\lambda_{\mathbf{L}}^+$.

Figure 2 shows a number of coupling graphs that have been used in our animation system. Panel a shows a symmetric chain of a set of N oscillators. In this case, the first nonzero eigenvalue of the matrix $\mathbf{L}_G$ can be shown to be $\lambda_{\mathbf{L}}^+ = 2(1-\cos(\pi/N))$ [WS05]. For a symmetric ring (panel b) one can show $\lambda_{\mathbf{L}}^+ = 2(1-\cos(2\pi/N))$.

Finally, a *star coupling* of $N > 2$ oscillators can be interpreted as a network, where $N-1$ oscillators are connected bidirectionally with the one in the center of the star, with the same weights, while they are not coupled with each-other. If the first oscillator is in the center this implies for the elements of the Laplacian matrix $(L_G)_{1,1} = N-1$, $(L_G)_{1,i} = (L_G)_{i,1} = -1$, $(L_G)_{i,i} = 1$, for $i > 1$, while all other entries are zero. It can be shown that the eigenvalues of this matrix are 0, 1 ($(N-2)$ times), and N. This implies $\lambda_{\mathbf{L}}^+ = 1$ and thus the stability condition $k > 1$.

4.5. Leader-group interaction

We will also consider scenarios where multiple characters are coupled asymmetrically to a single one, which can act as a 'leader' that controls the pattern of the others.

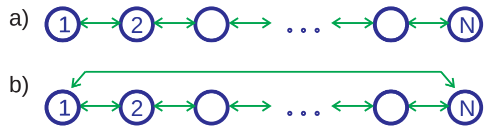

Figure 2: *Symmetric coupling. a) Chain and b) ring coupling of N oscillators.*

Assume first a follower scenario, where a single oscillator is coupled to the group of N identical oscillators that are already synchronized. The underlying dynamics is defined by:

$$\dot{\mathbf{x}}_0 = \mathbf{f}(\mathbf{x}_0) - k\left(N\mathbf{x}_0 - \sum_i \mathbf{x}_i\right) = \mathbf{f}(\mathbf{x}_0) - kN(\mathbf{x}_0 - \mathbf{x}_1)$$

A particular solution of this system is $\mathbf{x}_0 = \mathbf{x}_1$. If the system is partially contracting in $\mathbf{x}_0$ this implies the exponential convergence of the follower state $\mathbf{x}_0$ against the equilibrium state $\mathbf{x}_1$ of the other oscillators. This condition is obviously fulfilled if $kN > \sup_{\mathbf{x},t} \lambda_{\max}\left(\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)_s\right) = 1$.

In a leader scenario, the single oscillator feeds unidirectionally into all other N oscillators with the same coupling strength α, but not vice versa. This situation is described by the dynamics (for $1 \leq i \leq N$):

$$\begin{aligned} \dot{\mathbf{x}}_0 &= \mathbf{f}(\mathbf{x}_0) \\ \dot{\mathbf{x}}_i &= \mathbf{f}(\mathbf{x}_i) + k\mathbf{I}\sum_{j\in\mathcal{N}_i}(\mathbf{x}_j - \mathbf{x}_i) + \alpha(\mathbf{x}_0 - \mathbf{x}_i) \end{aligned} \quad (20)$$

Since the leader oscillator does not receive external inputs it oscillates autonomously, and $\mathbf{x}_0$ can be treated as external input. Applying partial contraction analysis to the second equation one obtains using the same definitions as in Section 3.3 and $\tilde{\mathbf{x}}_0 = [\mathbf{x}_0^T, ..., \mathbf{x}_0^T]^T$ the dynamics:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) - k\mathbf{L}\mathbf{x} - \alpha\mathbf{x} + \alpha\tilde{\mathbf{x}}_0 \quad (21)$$

This implies $\mathbf{J}(\mathbf{x},t) = \mathbf{D}(\mathbf{x},t) - k\mathbf{L} - \alpha\mathbf{I}$, and the contraction condition $\lambda_{\min}(k\mathbf{L}_G + \alpha\mathbf{I}) > \sup_{\mathbf{x},t} \lambda_{\max}\left(\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)_s\right) = 1$.

For the special case that the N oscillators (except for the leader oscillator) are symmetrically coupled all-to-all this contraction condition becomes $kN + \alpha > 1$. This implies that for $kN < 1$ contracting behavior can still be guaranteed when the coupling α to the leader oscillator is sufficiently strong. The minimum convergence rate is then given by $\rho_c = kN + \alpha$.

5. Results

5.1. Overview of the system

Our animation system is described in detail in [GMP*09]. It approximates periodic and non-periodic body movements by simple coupled networks of nonlinear dynamical systems (movement primitives) whose stable solutions are mapped onto the angle trajectories of a skeleton model with 17 joints. The mapping is learned using kernel methods from example trajectories that have been assessed using motion capture, and whose dimensionality is reduced applying a special blind source separation method that results in highly compact and accurate models for trajectories with very few estimated source terms. We have shown previously that this approach was suitable for the online generation of realistic-looking complex behavior of individual characters and for the simulation of group behavior, generated by coupling of the dynamical systems that control the individual avatars. Here, we exploit the advantage that the state dynamics can be chosen quite independently of the simulated behavior, making it possible to choose simple dynamical primitives so that the collective system dynamics is accessible for stability analysis.

An overview of the system is shown in Figure 3. The individual dynamic primitives, that control each individual character are given by Hopf oscillators. The stable solution of these oscillators are mapped onto three periodic 'source signals' $\sigma_j(t)$, $j = 1...3$ per character that are extracted from example trajectories by a special blind source decomposition algorithm that is described in detail in [OG06]. This algorithm approximates the joint angle trajectories $\xi_i(t)$ by a mixture model of the form:

$$\xi_i(t) = m_i + \sum_j w_{ij}\sigma_j(t - \tau_{ij}) \quad (22)$$

The mixing weights w_{ij}, time delays τ_{ij}, and the means of the joint angles m_i are determined by the blind source decomposition algorithm.

To generate the source signals online, the state of each limit cycle oscillator $[x,y]^T$ was mapped onto the values of the corresponding source signals σ_j using a *Support Vector Regression* (SVR) with Gaussian kernel [Vap98]. This mapping was trained with pairs of source signals extracted from motion capture data, and stable solutions of uncoupled Hopf oscillators.

The translation of the characters was computed by enforcing the kinematic constraints for the ground contact of the feet. Character speed was modulated by appropriate choice of the eigenfrequency ω of the oscillators.

5.2. Collective behavior fulfilling / violating contraction bounds

The following section presents a number of examples illustrating the behavior of groups of characters when the underlying dynamics fulfills or violates the bounds for contracting system behavior.

The first set of demonstrations shows the synchronization between a group of three characters with all-to-all coupling, for different coupling strengths. As shown in Section 4.1 for three oscillators in this case the dynamics is contracting for

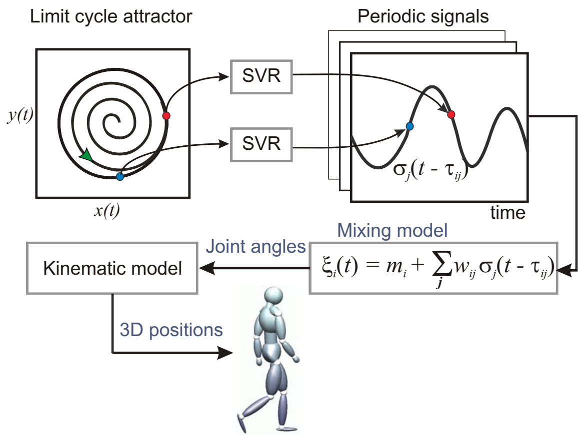

Figure 3: *Illustration of the system architecture for real-time animation. The phase space of the limit cycle oscillator is mapped onto the time-shifted source signals using Support Vector Regression. Joint angles are reconstructed by combining the time-shifted source signals linearly according to a learned mixture model.*

$k \geq 1/3$. [**Movie_1**] shows a group of characters, starting with random initial step phases, for the case that the coupling strength $k = 0.334$ fulfills this theoretical bound. In this case the dynamics quickly converges to a stable state, the characters walking in synchrony. Contrasting with this case, [**Movie_2**] shows an example where the coupling strength $k = 0.111$ violates the theoretical bound, resulting in very slow synchronization (reaching and equilibrium state only after hundreds of steps). The fact that the system still converges to a stable solution reflects that the bounds derived by contraction theory define sufficient, but not necessary stability conditions. For an even stronger violation of the theoretical bound, as shown in in [**Movie_3**] for the choice $k = -2.0$, results in a system dynamics that does not result in the formation of a coordinated behavioral pattern anymore. The strong coupling deforms the limit cycles in phase space, resulting in unnatural joint trajectories and very slow propagation of the characters.

The following set of demonstrations was generated assuming a bidirectional chain coupling between the oscillators. As shown in Section 4.1 for three oscillators (characters) in this case the dynamics is contracting for $k \geq 1$. [**Movie_4**] shows an example with $k = 1.0$ that fulfills the theoretical bound, resulting in the quick synchronization of the characters. Contrasting with this example, [**Movie_5**] shows the case $k = 0.333$ that violates the contraction condition. In this case the characters do not realize coordinated behavior in the observed time interval.

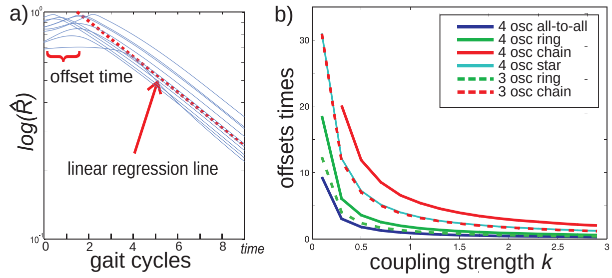

Figure 4: *a) Dispersion of the phase of the oscillators, averaged over 100 simulations with random initial conditions, as function of time (gait cycles). After an offset time, during which the dispersion remains relatively constant, it decays exponentially. Convergence rates were estimated by fitting linear function to this decay. b) Offset times (in gait cycles) as function of the coupling strength. (End of offset time interval was defined by the point where the regression line crosses the level $\hat{R} = 1$.)*

5.3. Theoretical vs. empirical convergence rates

As a more systematic validation of our theoretical bounds we also computed empirical convergence rates $\lambda^{\text{exper}} = 1/\tau^{\text{exper}}$ for groups of characters of different size N. These rates were obtained assuming approximately exponential convergence of the sizes of virtual displacements: $||\delta x|| \sim e^{-t/\tau^{\text{exper}}}$. The norm of the virtual displacements was approximated by the angular dispersion $\hat{R} = (1 - \frac{1}{N}|\sum_j e^{i\phi_j}|)^{\frac{1}{2}}$ [Kur84] of the phases ϕ_j of the oscillators, averaged over 100 simulations with random initial conditions.

Figure 4a) shows the logarithm of this dispersion measure as a function of time (in gait cycles). It shows an initial constant segment (offset time), and after that a nearly linear decay with time, from which the time constant τ^{exper} can be estimated by linear regression. Figure 4b) shows the offset times as function of the coupling strength to different types of coupling.

Figure 5a) shows the dependency between coupling strengths k and the convergence rate λ^{exper} as estimated from simulations in the regime of the exponential convergence. As derived from the theoretical bound, the convergence rate varies linearly in with coupling strength. In case of three oscillators the ring coupling is equivalent with all-to-all coupling. Figure 5b) shows the slope $d\lambda^{\text{exper}}(k)/dk$ of this linear relationship as function of N, the number of oscillators in the network. We find a close similarity between the theoretically predicted relationship (dashed curves) and the results from the simulation (indicated by the stars). In addition, it is evident that for all-to-all coupling the convergence rate increases with the number of oscillators, while for chain or ring coupling the convergence speed decreases with the number of oscillators (for fixed coupling strength). These results show in particular that the proposed theoretical frame-

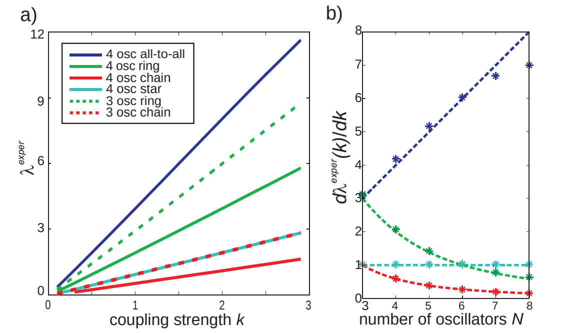

Figure 5: *a) The relationship between convergence rate and coupling strength k for different types of coupling graphs. b) Slopes of this relationship as function of the number N of Hopf oscillators, comparing simulation results (indicated by asterisks) and derived from the theoretical bounds (Section 4.1).*

work is not only suitable for proving asymptotic stability, but also for guaranteeing the convergence speed of the system dynamics.

5.4. Leader-group interaction

As discussed in Section 4.5, one can introduce a leader that can entrain all other characters in the scene by its own behavior. In addition, coupling with a leader can synchronize other characters in the scene that would not synchronize otherwise. In Section 4.5 we showed that, assuming k signifies the coupling strength between the members of the group and α the strength of the coupling between the members and the leader, contracting behavior is obtained for $kN + \alpha > 1$.

Different behaviors are illustrated in the following movies, that show five characters. One of them is the leader (dark grey) that was coupled unidirectionally to all members of the of group. Without the leader the group ($\alpha = 0$) shows exponential convergence for $k > 1/4$. The movie [**Movie_6**] shows a case with $k = 0.01$, i.e., the system is non-contracting and no coordinated behavior is reached in the simulation. If a leader with sufficiently strong coupling to the other group members ($\alpha = 1$) is introduced the theoretical contraction bound is fulfilled. As shown in [**Movie_7**], in this case fast convergence to a coordinated behavior is observed even for small values of k. The next example shown in [**Movie_8**] corresponds to the case $k = 0.2$ and $\alpha = 0.25$, fulfilling also the theoretical bound for contraction. The characters converge very quickly to a coordinated behavior. This case shows an example where all characters are initially not coupled (yellow bar on time line) and start with random initial phases. After activation of the coupling (green bar) the leader experiences a phase perturbation. The group quickly adopts again the behavior of the leader. [**Movie_9**] shows a corresponding example where a perturbation of the same size is not applied to the state of the leader but to the one of a group member. In this case, the group member quickly adopts again the behavior of the group. (All movies can be also found [**here**].)

6. Conclusion

We have presented a new approach for the design stability properties of animation systems that generate collective behavior of characters through self-organization from interacting dynamical models. Opposed to many existing approaches, our system generates realistic-looking behavior from relatively simple nonlinear dynamical systems. Opposed to more detailed biomechanical or physical models, such systems are in principle accessible for an application of tools from stability theory. We also have introduced contraction analysis as a new framework for the study and to design the stability properties of online animation systems. The proposed theoretical approach has the advantage that it permits to derive global stability results for nonlinear systems from local stability properties that can be easily verified. In addition, contraction theory offers the possibility to treat the stability of complex systems, since it permits to transfer stability results from components to composite systems [LS1998]. Many other approaches for stability analysis do not have this property, which makes the analysis of complex systems often intractable. In addition, we demonstrated that this theory also is suitable to compute bounds for the convergence rate of such systems, where sufficiently fast convergence is important for many applications.

The examples presented in this paper were very simple, and we have presented elsewhere [GMP*09, PMOG08, MPO*08] that the same type of animation system also can simulate much more complex and non-periodic body movements. While the present paper tried to sketch some first steps towards a development of a systematic approach for the design of the dynamics of such online animation systems, future work needs to test whether this work can be extended to more complex systems that combine periodic and non-periodic primitives, and other dynamical components, such as navigation, or the arbitration of different behaviors.

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