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A Study Of Discretization Errors In Volume Rendering Integral Approximations

Tiago Etiene[1], Robert M. Kirby[1] and Cláudio T. Silva[2]

[1] School of Computing, University of Utah
[2] Poly-NYU

Abstract

We present a study of the influence of different approximation schemes on the convergence rate of volume rendering integral (VRI) numerical approximations. We experimentally evaluate the impact of numerical integration techniques on the rate of convergence to the correct solution of the VRI on a single ray. We report that the discretization of both the inner and outer integrals have influence on the overall convergence rate. Then, we present results related to the (traditional) pre-integrated and second-order pre-integrated algorithms. In practice, we observed that pre-integrated lookup tables provide second and third order convergence rates for the VRI approximation, respectively. Our results also suggest that the convergence rate drops one order of magnitude for the second-order algorithm when lookup tables are numerically computed using low sample rates. Also, the convergence of both algorithms drops to linear when the attenuation within ray segment is neglected.

1. Introduction

The volume rendering literature covers many topics related to approximation errors of the VRI; these include studies on the bounds on the magnitude of approximation error [NA92], interpolant accuracy [EHMDM08, MMMY96, MMMY97], errors due to limited precision [KUMY10], and others [WM92, WMS98]. These studies are specially important in areas where the image quality plays a crucial role, such as medical imaging [PH02]. In this work, we focus on the *order of accuracy* of the numerical solutions of the VRI, which, to the best of our knowledge, has not been studied. The order of accuracy is the rate at which a numerical approximation converges to the correct solution when the number of samples increases ($n \to \infty$), or, equivalently, the sample spacing vanishes ($d \to 0$). Etiene *et al.* [EJR*13] provides the theoretical analysis of the discretization errors of the VRI approximations that we will use here. We measure how the image quality (errors) changes as a function of n and the numerical integration techniques used. Our contributions are the following: we study how different numerical integration techniques affect the errors of the approximated solution to the VRI using both Newton-Cotes formulas and pre-integrated volume rendering.

2. Discretization errors

We assume the low albedo emission plus absorption model [Max95]. The volume rendering integral I is:

$$I = \int_0^D C(s(\lambda))\tau(s(\lambda)) \exp\left(-\int_0^\lambda \tau(s(\lambda'))\mathrm{d}\lambda'\right)\mathrm{d}\lambda, \quad (1)$$

where D, $C(s(\lambda))$, $\tau(s(\lambda))$, and $s(\lambda)$ are the ray length, the emitted light, the extinction coefficient, and the scalar value at position λ respectively. In this paper, we assume that τ and C have bounded derivatives of all orders.

In this section, we provide a corollary extension to the derivation proposed by Etiene *et al.* [EJR*13]. In previous work, the authors were interested in verification of the approximations made in traditional volume rendering systems; with respect to the VRI, they focused on errors derived from the rectangle method of integration. In this work, we focus on only the VRI approximation and provide an extension to the derivations in Etiene *et al.*: 1) we use Newton-Cotes formulas for the numerical integration, instead of the particular case of the Riemann summation and 2) we *do not* discretize $\exp(\cdot)$ in Equation (1). The discretization version of the VRI is given by [EJR*13]:

$$I = \sum_{i=0}^{n-1} C_i \tau_i d \prod_{j=0}^{i-1} (1 - \tau_j d) + O(d), \quad (2)$$

where i is the i-th sample point in the ray, $C_i = C(s(id))$, $\tau_i = \tau(s(id))$, n is the number of ray samples, and $d = D/n$. Equation (2) shows that errors decay linearly with respect to d when Riemann summation is used for both the inner and outer integrals. Let us revisit the discretization errors of the inner integral. Our derivation assumes that the integration method being used has the following form:

$$t(\lambda) = \int_0^\lambda \tau(\lambda')\mathrm{d}\lambda' = \sum_{\substack{j=0 \\ j \leftarrow j+a}}^{i-1} \sum_{k=j}^{a} w_k(d)\tau_k + O(d^p), \quad (3)$$

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where $w_p(d)$ is a linear function of d, $\lambda = id$, a is the number of sample points used by the integration method minus one, and p is the order of accuracy of the integration method over the interval $[0, D]$, not within a ray segment. This is not too restrictive, given that many integration methods can be written in this form, in particular, the Newton-Cotes formulas. The notation $j \leftarrow j + a$ means that a samples should be skipped due to the integration method used. Henceforth, we implicitly skip samples and thus omit the use of $j \leftarrow j + a$ for the sake of notational clarity. We further assume that $a \ll n$ because approximation errors should be written as a function of sample distance d instead of the domain interval D. In practice, this is a reasonable assumption. We assume that the inner and outer integrals in Equation (1) can be discretized using different accuracy degrees. As an example, if Simpson's Rule is used, Equation (3) becomes:

$$t(\lambda) = \sum_{j=0}^{i-1} \left(\frac{d}{3}\tau_j + \frac{4d}{3}\tau_{j+1} + \frac{d}{3}\tau_{j+2} \right) + O(d^4). \quad (4)$$

Substituting Equation (3) into $\exp(\cdot)$ in Equation (1):

$$T(\lambda) = \exp\left(-\sum_{j=0}^{i-1} \sum_{k=j}^{a} w_k(d)\tau_k + O(d^p) \right) \quad (5)$$

$$= \exp(O(d^p)) \prod_{j=0}^{i-1} \exp\left(-\sum_{k=j}^{a} w_k(d)\tau_k \right). \quad (6)$$

The discretization errors can be trivially retrieved for a single ray segment ($i = 1$). Because errors accumulate as the ray is traversed, we focus on the general case $i = n$. The next step is to discretize $\exp(x)$. In practice, many implementations use a linear approximation, *i.e.* $\exp(x) = 1 - x + O(x^2)$, $x \to 0$. We, on the other hand, assume that no error is introduced at this step and the exponential is computed exactly. This is not what is typically done in volume rendering implementation, but it will suffice to illustrate how the different error sources affects the overall convergence rate. Thus, we can further expand the previous equation:

$$T(\lambda) = (1 + O(d^p)) \prod_{j=0}^{n-1} \exp\left(-\sum_{k=j}^{a} w_k(d)\tau_k \right) \quad (7)$$

$$= \prod_{j=0}^{n-1} \exp\left(-\sum_{k=j}^{a} w_k(d)\tau_k \right) + \quad (8)$$

$$O(d^p) \prod_{j=0}^{n-1} \exp\left(-\sum_{k=j}^{a} w_k(d)\tau_k \right). \quad (9)$$

Because $w_k = O(d)$, τ and its derivatives are bounded, and $a \ll n$, we can write $\sum_{k=j}^{a} w_k(d)\tau_k = O(d)$:

$$T(\lambda) = \prod_{j=0}^{n-1} \exp\left(-\sum_{k=j}^{a} w_k(d)\tau_k \right) + O(d^p) \prod_{j=0}^{n-1} \exp(O(d)). \quad (10)$$

Since $n = D/d$, the second term simplifies to:

$$O(d^p) \prod_{j=0}^{n-1} \exp(O(d)) = O(d^p) \exp\left(\sum_{j=0}^{n-1} O(d) \right) \quad (11)$$

$$= O(d^p) \exp(nO(d)) \quad (12)$$

$$= O(d^p) O(1) = O(d^p). \quad (13)$$

Plugging Equation (13) into Equation (10):

$$T(\lambda) = \prod_{j=0}^{n-1} \exp\left(-\sum_{k=j}^{a} w_p(d)\tau_p \right) + O(d^p) \quad (14)$$

$$= \tilde{T}(\lambda) + O(d^p), \quad (15)$$

where $\tilde{T}(\lambda)$ is the approximation of $T(\lambda)$. The discretization of the outer integral is also based on Equation (3) but assuming an error of order $O(d^q)$:

$$I = \sum_{i=0}^{n-1} \sum_{k=i}^{b} \phi_k(d) C_k \tau_k T_k + O(d^q) \quad (16)$$

$$= \sum_{i=0}^{n-1} \sum_{k=i}^{b} \phi_k(d) C_k \tau_k \left(\tilde{T}_k + O(d^p) \right) + O(d^q) \quad (17)$$

$$= \sum_{i=0}^{n-1} \sum_{k=i}^{b} \phi_k(d) C_k \tau_k \tilde{T}_k + \quad (18)$$

$$O(d^p) \sum_{i=0}^{n-1} \sum_{k=i}^{b} \phi_k(d) C_k \tau_k + O(d^q), \quad (19)$$

where $\tilde{T}(\lambda) = \tilde{T}(id) = \tilde{T}_i$. Since $\sum_{k=i}^{b} \phi_k(d) C_k \tau_k = O(d)$:

$$O(d^p) \sum_{i=0}^{n-1} \sum_{k=i}^{b} \phi_k(d) C_k \tau_k = O(d^p) \sum_{i=0}^{n-1} O(d) \quad (20)$$

$$= O(d^p) nO(d) = O(d^p). \quad (21)$$

Substituting Equation (21) into Equation (19):

$$I = \sum_{i=0}^{n-1} \sum_{k=i}^{b} \phi_k(d) C_k \tau_k \tilde{T}_k + O(d^p) + O(d^q). \quad (22)$$

Equation (22) states that the dominant error in the numerical approximation of the VRI is $r = \min(p, q)$. This means that the methods used for discretizing both the inner and outer integrals have influence on the approximation of the VRI. In this work, we have ignored the influence of the approximation errors arising from the discretization of $\exp(\cdot)$.

3. Numerical experiments

Table 1 shows the order of accuracy of the VRI discretization using several numerical integration techniques. Rows and columns represent the method used for the discretization of the inner and outer integrals, respectively. Each entry of the table shows the obtained order of accuracy for the VRI discretization. We use an analytical solution to measure the difference between the correct and approximated solution as $d \to 0$ [EJR*13]. The experiments shown in this section use: $D = 1$, $s(\lambda) = \lambda$, $C(s) = \sin(s^2)$, $\tau(s) = s\cos(s^2)$,

and the analytical solution for this case is $I = 2 - (\sin(D^2) + 2)\exp(-\sin(D^2)/2)$. It works as follows: i) we start by sampling a ray segment in the interval $[0,1]$ at $n = 5$ points and progressively increases the sampling level by doubling the number of points ($n = 2n - 1$ or equivalently $d = d/2$); ii) for each level, we assign a scalar value according to s and use the transfer functions τ and C to approximate the VRI; iii) we then compute the absolute errors between the numerical and analytical solution. Given the errors for each sampling level, one can compute how fast the error converges towards zero (the rate k shown in Table 1). For details on order of accuracy method, we refer the reader to Babuska and Oden [BO04].

We run our experiments not only with Newton-Cotes formulas, but also with other methods (marked with an *). For non-Newton-Cotes formulas, the error expansion involves different bounds on the integrals, and function evaluations at points non-aligned with the ray samples points. Thus, these methods do not directly fit into the derivation shown previously. We use the following methods for the inner integral:

I $t = \tilde{t} + O(d)$: Riemann summation;
II $t = \tilde{t} + O(d^2)$: Trapezoidal Rule;
III $t = \tilde{t} + O(d^4)$: Simpson's Rule;
IV $t = \tilde{t} + O(1/\sqrt{n})$: Monte-Carlo Integration;*
V $t = \tilde{t} + O(d^6)$: Gauss Quadrature (3 points).*

And we used the following integrators for the outer integral:

i $I = \tilde{I} + O(d)$: Riemann summation;
ii $I = \tilde{I} + O(d^2)$: Trapezoidal Rule;
iii $I = \tilde{I} + O(d^4)$: Simpson's Rule;
iv $I = \tilde{I} + O(d^6)$: Boole.*

Table 1 (top) shows the interplay between discretization errors from the inner and outer integral described in our previous section. As the results suggest, to increase the overall accuracy of the method, both the inner and outer integral must be improved. In practice, it is common to increase the quality of the numerical integration by simply increasing the number of samples. Our experiments shows *how much improvement* one should expect assuming different degrees of approximation.

The bottom table presents a slightly different scenario. Although our analysis does not include the discretization of the exponential term, one can still run convergence tests and observe outcomes. In this experiment, we use a linear approximation of the exponential term, $\exp(x) = 1 - x + O(x^2)$, and evaluate the convergence of the VRI numerical approximation. The results reveal that the error convergence is approximately linear for all but Monte-Carlo method. By using, on the other hand, a cubic approximation and Trapezoid method for the inner and outer integral, we experimentally obtained $O(d^{2.00})$ error. Clearly, not only the inner and outer integral have influence over the VRI approximation error but also the discretization of $\exp(\cdot)$ in Equation (1).

Table 1: *The table shows the convergence rates $O(d^k)$ (only k is shown) of the VRI approximation. As a rule of thumb, high values of k are better. Rows and columns represent the discretization method used for inner and outer integral respectively. The first letter of the name of the each method is shown for clarity. The top table does not approximate $\exp(x)$ whereas the bottom table uses a linear approximation ($\exp(x) = 1 - x + O(x^2)$).*

	R i	T ii	S iii	B* iv
R I	1.01	1.00	1.00	1.00
T II	1.00	2.00	1.98	2.00
S III	0.99	1.99	4.02	4.80
M* IV	0.48	0.41	0.42	0.42
G* V	1.00	2.00	4.00	5.54

	R i	T ii	S iii	B* iv
R I	1.03	1.00	1.01	1.01
T II	0.99	1.06	0.98	0.98
S III	0.99	1.08	1.01	1.01
M* IV	0.08	0.01	0.00	0.04
G* V	0.99	1.08	1.01	1.01

4. Pre-integrated volume rendering

Pre-integrated volume rendering separates the computation of the VRI into the numerical integration of C and τ, and the approximation of the scalar field s. As a result, pre-integration can deal with high-frequencies and generate high-quality images. The idea is to pre-integrate C and τ between consecutive sample points i and $i+1$. The precomputed values can be retrieved by using a lookup table indexed by s_f and s_b, the scalar values at consecutive sample points. The pre-integrated values C_i and α_i are:

$$\alpha_i = \exp\left(-\int_{id}^{(i+1)d} \tau(s(\lambda))d\lambda\right) \quad (23)$$

$$C_i = \int_{id}^{(i+1)d} C(\lambda)\tau(\lambda)\exp\left(-\int_{id}^{\lambda} \tau(\lambda)d\lambda'\right)d\lambda \quad (24)$$

In this section, we focus on two ways of pre-integrating the previous equations: using linear [EKE01] and quadratic [EHMDM08] interpolation of the scalar field. In our experiments, we use the following configuration for convergence analysis: $D = 1$, $s(\lambda) = 1.0/(1+\lambda)$, $\tau(s) = s$, $C(s) = 1/s$, $I = \log(1+D)$. Our lookup table is composed by the analytical solutions of α_i (using linear and quadratic interpolants) and C_i (using linear interpolant). No analytical solution is available for C_i using quadratic interpolant. In this case we numerically integrate C_i using $N = 10$ and $N = 100$ sample points per ray segments. As in previous section, the convergence is written in terms of the sample distance $d = D/n$; Table 2 shows the obtained results. As before, we assume the numerical integration of a single ray. The best result is

the expected one: using a quadratic interpolation for both C_i ($N = 100$) and α_i, a $O(d^{2.99})$ error is obtained. Linear interpolation provides $O(d^2)$ error. Note that these results are better than the typical VRI discretization. Also, when using quadratic approximation, the quality of the numerical integration of C_i affects the convergence rate: $N = 100$ result in $O(d^{2.99})$ whereas $N = 10$ leads to $O(d^{1.80})$. Finally, by neglecting the ray attenuation within the segment, one can accelerate table computation but at expense of the order of accuracy (shown in the last row).

We now provide a sketch of the convergence analysis for the pre-integrated algorithm. This preliminary analysis provides useful insights to understand the results shown in Table 2. Let us focus on α_i. A linear interpolation of the scalar field between adjacent sample points provides a 2nd order error accuracy between consecutive sample points: $s(\lambda) = s_f + \frac{\lambda - id}{d}(s_b - s_f) + O(d^2) = \tilde{s} + O(d^2)$. We expand τ using Taylor series: $\tau(\tilde{s} + O(d^2)) = \tau(\tilde{s}) + O(d^2)$. Thus:

$$\alpha_i = \exp\left(-\int_{id}^{(i+1)d} \tau(\tilde{s}) + O(d^2)\mathrm{d}\lambda\right) \quad (25)$$

$$= \exp\left(-\int_{id}^{(i+1)d} \tau(\tilde{s})\mathrm{d}\lambda + O(d^3)\right) \quad (26)$$

$$= \exp\left(-\int_{id}^{(i+1)d} \tau(\tilde{s})\mathrm{d}\lambda\right)\exp(O(d^3)) \quad (27)$$

$$= \exp\left(-\int_{id}^{(i+1)d} \tau(\tilde{s})\mathrm{d}\lambda\right) + O(d^3). \quad (28)$$

The error for the ray segment $[id, (i+1)d]$ is $O(d^3)$, so we expect it to drop to $O(d^2)$ when integrating it over $n = D/d$ sample points, which means that α_i approximation error is second-order. Using similar steps, one can show that C_i is also $O(d^3)$ within a segment and we expect the order of of magnitude to drop as we integrate over the entire ray. The steps shown above can be used with other interpolants. A second-order Lagrange polynomial yield $O(d^3)$ approximation error for the scalar field, which matches the results obtained in Table 2.

The convergence rate may also affected by other approximations (see bottom row in Table 2). Engel *et al.* [EKE01] suggest a simplification for C_i to improve the performance of the lookup table computation. The simplification ignores attenuation within the i-th ray-segment. In practice, the obtained convergence drops to $O(d)$. Mathematically, this is equivalent to use a constant approximation within $[id, (i+1)d]$. By setting $\exp(x) = 1 + O(x)$ in Equation (24):

$$C_i = \int_{id}^{(i+1)d} C(\lambda)\tau(\lambda)\left(1 + O\left(\int_{id}^{\lambda}\tau(\lambda')\mathrm{d}\lambda'\right)\right)\mathrm{d}\lambda. \quad (29)$$

Assuming that τ is bounded:

$$= \int_{id}^{(i+1)d} C(\lambda)\tau(\lambda)(1 + O(d))\mathrm{d}\lambda \quad (30)$$

$$= \int_{id}^{(i+1)d} C(\lambda)\tau(\lambda)\mathrm{d}\lambda + O(d^2). \quad (31)$$

Table 2: *The table shows the convergence rates $O(d^k)$ (only k is shown) of the VRI approximation. As a rule of thumb, high values of k are better. The first row shows the approximation of the pre-integrated C_i using linear interpolation. The second and third rows show the results of applying a second-order approximation (C_i is numerically integrated using $N = 100$ and $N = 10$ sub-intervals respectively). The last row shows the convergence obtained by ignoring attenuation within the i-th ray segment [EKE01]. The columns represent the linear and quadratic approximation of the scalar field for α_i.*

	Lin.	Quad.
Lin.	2.00	1.99
Quad. (N=100)	1.98	2.99
Quad. (N=10)	1.98	1.80
Approx.	1.00	1.01

The last equation is the simplification proposed to accelerate the lookup table computation. Again, by integrating the previous equation over n samples, the error convergence rate becomes linear. The error produced within the segment is $O(d^2)$, instead of $O(d^3)$. Note that the $O(d^2)$ error will not change even when high-order interpolants are used.

5. Conclusion and Future Work

In this work, we have shown the impacts of both the inner and outer integral over the convergence of the VRI approximation. Using pre-integrated algorithm, we have shown that the high convergence rate obtained from the method drops for the performance optimizations presented.

This work can be expanded in several ways: first, we believe that a thorough theoretical analysis of pre-integrated algorithms in needed; more error sources should be added to the standard discretization of the VRI, including error due to the scalar field s and $\exp(\cdot)$; our work and previous work assume as input smooth functions when in practice they may not even be C^0. Another potentially interesting research direction is to study the approximation errors of non-smooth function.

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