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Polynomial Approximation of Blinn-Phong Model

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Abstract

The Phong model has been one of the oldest and the most popular reflection models in Computer Graphics. It can be used to model specular highlights of various materials. In this paper, we consider a polynomial model and obtain a linear approximation of the Blinn-Phong model. Approximation errors were obtained for the proposed model and empirical comparisons were made using a measured BRDF data set. Based on the empirical results, it is shown that proposed model provides visually convincing representation of BRDF and performs well for modeling the surface reflectance.

Categories and Subject Descriptors (according to ACM CCS): I.3.7 [Computer Graphics]: Three-Dimensional Graphics and Realism

1. Introduction

Building a reflection model for generating photo realistic images is an important problem in computer graphics. Finding an appropriate model for the complete description of light and material interaction is essential for achieving a realistic simulation of surface reflectance. A simple way of making such a simulation is to use physical reflectance measurements. These measurements can be used to estimate parameters of the underlying model [MWL00].

A class of functions called *Bidirectional Reflectance Distribution Functions* (BRDFs); have been used to model the relationship between the incoming and outgoing radiance. At a given surface point the amount of light reflected in the outgoing direction is a function of incoming light and the BRDF. For a single point light source the outgoing radiance is given by

$$L(\boldsymbol{\omega}_o) = f(\boldsymbol{\omega}_o, \boldsymbol{\omega}_i) L(\boldsymbol{\omega}_i)(\mathbf{n} \cdot \boldsymbol{\omega}_i) \quad (1)$$

where $\boldsymbol{\omega}_i$ and $\boldsymbol{\omega}_o$ are unit vectors of incoming and outgoing directions respectively, $\mathbf{n}$ is the surface normal and f is the BRDF [AH02].

Various models have been proposed to represent BRDFs. Very popular among these models has been the Phong model [Pho75] which was originally introduced to model the effect of highlight in computer generated images. The specular term for the Phong model may be written as

$$I_{spec} = \sigma(\mathbf{r} \cdot \mathbf{v})^{\gamma} \quad (2)$$

where σ is specular coefficient, γ is surface dependent constant that controls the sharpness of the specularity, $\mathbf{v}$ is the vector from the surface point to the viewer, and $\mathbf{r}$ is the reflection of incoming light vector $\mathbf{l}$ through normal vector $\mathbf{n}$ of the underlying surface. Using the angle between the normal vector and the halfway vector

$$\mathbf{h} = \frac{\mathbf{l} + \mathbf{v}}{\|\mathbf{l} + \mathbf{v}\|} \quad (3)$$

Blinn [Bli77] introduced the following variation of the original Phong Model

$$I_{spec} = \sigma(\mathbf{n} \cdot \mathbf{h})^{\gamma}, \quad \gamma > 0 \quad (4)$$

This model is computationally more convenient than the model in (1) since it does not require computation of the reflection vector **r.**

Based on the equation (4) the relationship between the Blinn's form of Phong highlighting model and the general BRDF lighting equation can be written as [Wyn00]

$$f(\boldsymbol{\omega}_o, \boldsymbol{\omega}_i) = \frac{\sigma(\mathbf{n} \cdot \mathbf{h})^{\gamma}}{\mathbf{n} \cdot \boldsymbol{\omega}_i} \quad (5)$$

This model neither obeys the reciprocity nor the energy conserving properties of the BRDFs [Kau04].

When the BRDF function in (5) is substituted in (1), the term $\mathbf{n} \cdot \boldsymbol{\omega}$ that is the cosine of the angle between the normal and incoming light direction cancels out. Therefore the Blinn-Phong BRDF function in (5) essentially is a non-linear function of ($\mathbf{n} \cdot \mathbf{h}$) only. For a given set of BRDF measurements of a certain material the corresponding parameters of the model can be estimated by using the least squares technique. However finding the optimal solution is not straightforward and non-linear least squares estimation requires using some optimization algorithms. The underlying estimation procedure suffers from stability problems because the algorithm may locate a local minimum which may not a global one and it depends on the choice of good initial values.

In this paper, we approximate the Blinn-Phong model by a polynomial in single variable $x = \mathbf{n} \cdot \mathbf{h}$. The resulting model is linear in parameters and corresponding parameters can be estimated using standard multiple regression techniques. The proposed model avoids most of the problems associated with parameter estimation with the hope of better representation of BRDF measurements.

In the next section we introduce the polynomial model as an approximation of the Blinn-Phong model, explain corresponding parameter estimation procedure and investigate the error introduced by polynomial approximation when the true model is assumed to be the Blinn-Phong model. Empirical results are presented in section 3. Section 4 is devoted for conclusions.

2. Approximation of the Blinn-Phong model

The Blinn-Phong BRDF model which is based on the sum of a diffuse term and a specular term can be written as

$$f = \mu + \sigma(\mathbf{n} \cdot \mathbf{h})^{\gamma} \tag{6}$$

where μ stands for the diffuse term and the second term is defined as in (4). The parameters μ and σ are linear while γ is non-linear. Although the above model offers a simple functional form for representing BRDFs it encounters some problems in estimating the non-linear parameter γ . We proceed to approximate this model by employing a linear model. The proposed model is a polynomial of the form

$$f = \beta_0 + \beta_1 x + \beta_2 x^2 + \cdots + \beta_p x^p \tag{7}$$

where $\beta_0, \beta_1, \beta_2, \ldots, \beta_p$ are the linear parameters to be estimated, p is the degree of the polynomial and $x = \mathbf{n} \cdot \mathbf{h}$. The parameter β_0 corresponds to the diffuse term μ in the Blinn-Phong model. The accuracy of the proposed model depends on the choice of the degree of the polynomial. Naturally, approximation can be improved with the inclusion of additional terms in the model at the expense of creating some storage and computational problems in real life applications. Based on the predefined level of approximation error, the corresponding degree of the polynomial can be determined in advance or by visual inspection of the images obtained from BRDF data. Approximations of the Blinn-Phong model with parameters $\mu = 0, \sigma = 1, \gamma = 20$ by polynomials of various degrees are illustrated in Figure1. It is seen from the figure that the differences between the fitted and true models almost indistinguishable when $p \geq 6$. For polynomials of lower degrees, the corresponding maximum errors are observed at x=1.

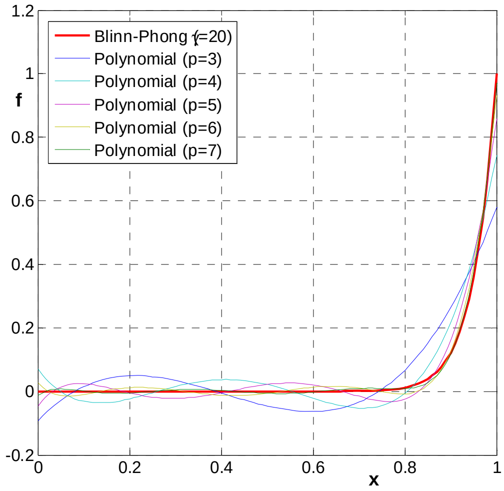

Figure 1. Approximations of the Blinn-Phong Model by polynomial models ($\mu = 0, \sigma = 1, \gamma = 20$).

2.1. Parameter estimation

Parameters of the Blinn-Phong model can be estimated by using non-linear least squares technique. One major problem with the estimation has been that the variances of BRDF measurements are not homogeneous as the BRDF measurements obtained at grazing angles are prone to outliers [NDM05]. Appropriate weights must be used to stabilize variances and to reduce outlier effects on the parameter estimates.

For a fixed value of $\gamma = \gamma_0$, the weighted least squares estimates of μ and σ can be obtained as

$$\begin{aligned} \hat{\mu} &= \left(\sum_{i=1}^{n} w_i (f_i - \hat{\sigma} x^{\gamma_0}{}_i \right) / \sum_{i=1}^{n} w_i \\ \hat{\sigma} &= \left(\sum_{i=1}^{n} w_i x_i (f_i - \hat{\mu} \right) / \sum_{i=1}^{n} w_i x_i^{2\gamma_0} \end{aligned} \tag{8}$$

where w_i and f_i, $(i = 1, 2, \ldots, n)$ are the weights and the measured BRDFs respectively, and $x = \mathbf{n} \cdot \mathbf{h}$. Using these results, the objective function for estimating the parameter γ can be expressed as

$$\varphi(\gamma) = \sum_{i=1}^{n} w_i (f_i - \hat{\mu} - \hat{\sigma} x_i^{\gamma})^2 \quad (9)$$

Non-linear optimization algorithms may be used to obtain the estimate of γ.

Weighted least squares estimates of the parameters of the polynomial model in (7) can be obtained by a standard linear multiple regression technique. However, in our previous study [OKB06] we proposed using robust least squares approach for estimating parameters of multi-variable polynomial models. In this study also we use a robust regression algorithm called bisquare weights. An advantage of using this approach is that the weights allocated to each data point depend on how far the observation is from the fitted curve. An outlier is given a zero weight while a point close to curve gets a full weight.

2.2. Approximation error

Obviously, the representational ability of a polynomial model is not bounded by the Blinn-Phong model. However, the bias of the approximation may be of interest in some practical applications when the true model is assumed to be a Blinn-Phong model.

The least squares estimate of the polynomial model in (7) can be expressed in matrix notation as

$$\hat{f} = \mathbf{X}\hat{\boldsymbol{\beta}} \quad (10)$$

where $\mathbf{X}$ is an $n \times (p+1)$ matrix whose ith row is $\mathbf{x}_i' = (1\ x_i\ x_i^2\ \cdots\ x_i^p)$ and $\hat{\boldsymbol{\beta}}' = (\beta_0\ \beta_1\ \cdots\ \beta_p)$. Assuming that the true model is the Blinn-Phong model that is $\eta = \mu + \sigma x^{\gamma}$ then the square of the bias term corresponding to the ith observation for the model in (10) can be written as [OKB06]

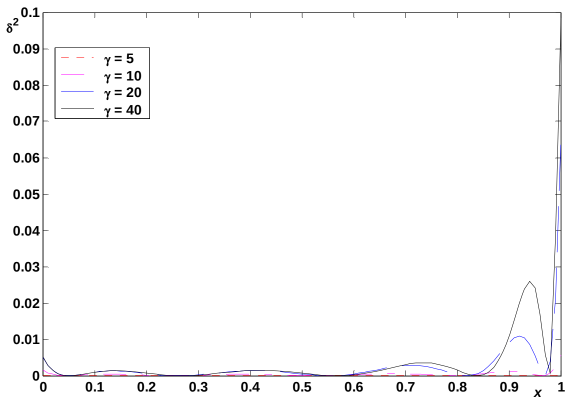

Figure 2. *Approximation error of the fitted polynomial model of degree p=4 when the true model is the Blinn-Phong Model ($\mu = 0, \sigma = 1$).*

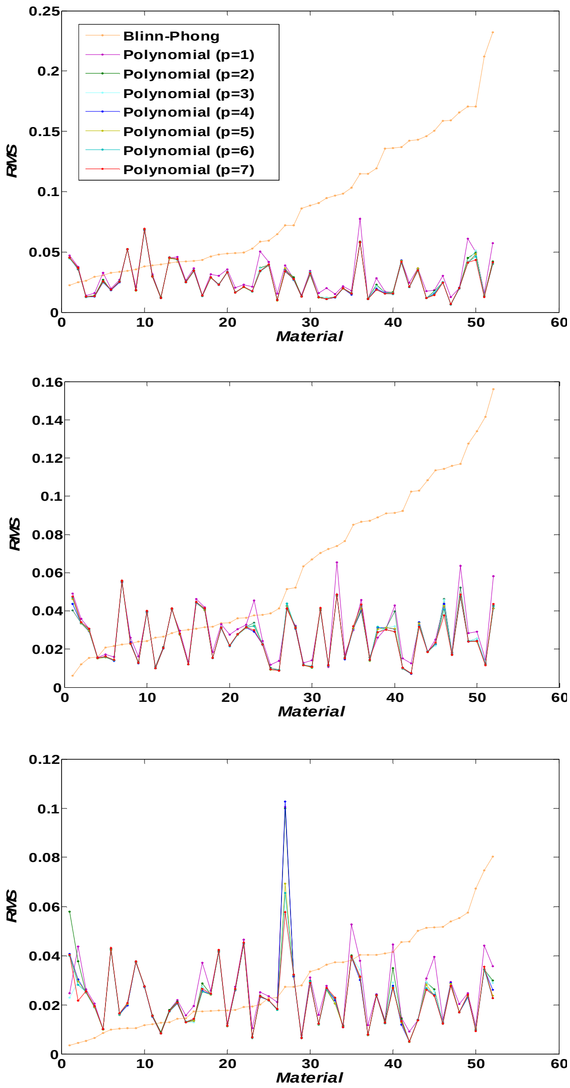

Figure 3. *Root mean squares of the fitted Blinn-Phong model and the polynomial models of degree p=1, 2, …, 7. Top: Red, middle: Green, Bottom: Blue.*

$$\delta_i^2 = \left\{ \left[\mathbf{x}_i' (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}\, \mathbf{x}^{(\gamma)} - \mathbf{x}_i^{(\gamma)} \right] \boldsymbol{\alpha} \right\}^2 \quad (11)$$

where $\mathbf{X}$ and $\mathbf{x}_i$ are defined as in (10), $\mathbf{x}^{(\gamma)}$ is an $n \times 2$ matrix whose ith row is $\mathbf{x}_i^{(\gamma)'} = (1\ x_i^{\gamma})$ and $\boldsymbol{\alpha} = (\mu\ \ \sigma)'$.

Note that the bias term depends on the parameters of the true model and the design matrix **X.**

Since the fitted and the true models are univariate functions, the bias term essentially is a function of x only. Graphs of the bias term as a function of x for $\mu = 0$, $\sigma = 1$ and γ =5, 10, 20, 40 when the fitted model is a polynomial of degree 4 are shown in Figure 2. These results compare with those of Figure 1.

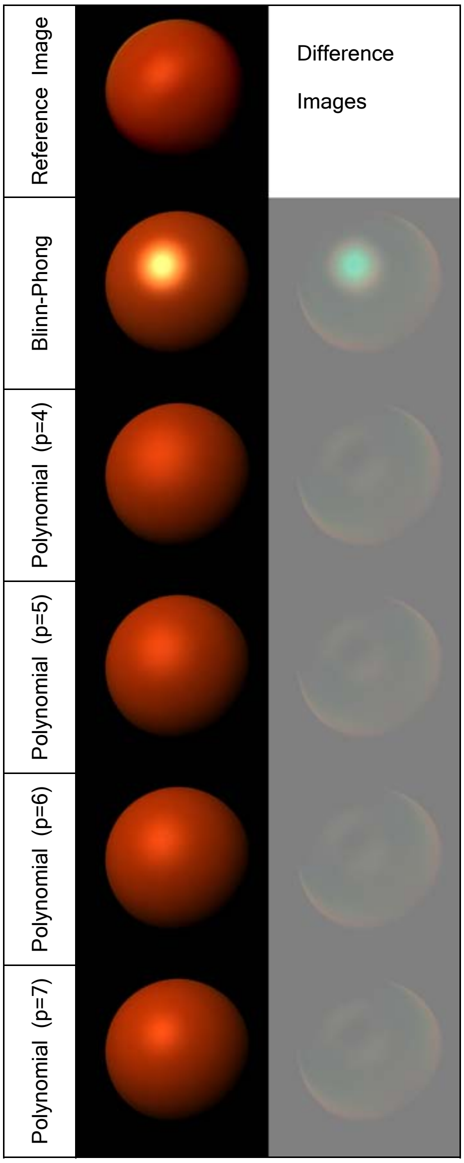

Figure 4. *Comparison of the spheres obtained by using Zernike polynomials (top) and, the Blinn-Phong and polynomial models (Orange peel). Difference images were obtained by subtracting the reference image from those based on the corresponding models.*

3. Results

To conduct further evaluation of the proposed approach one needs to investigate how well the polynomial approximation performs on the measured BRDF data. In this study we have used the Columbia Utrecht Reflectance and Texture (CUReT) database which contains BRDFs of a variety of real world surface materials [CURe]. In this database, 205 BRDF measurements are provided for each of the sample material.

A fair comparison scheme of the models should consider the real images and the generated images based on the polynomial approximation and the Blinn-Phong model. Using 205 BRDF measurements for each sample material we created test images by fitting Zernike polynomials of order 8 [KvDS96]. Images of spheres obtained using the polynomial models and the Blinn-Phong model are compared to that of the Zernike polynomials.

Figure 3 shows the plots of the root mean squared errors based on the Blinn-Phong model and polynomial models of various degrees. Results are obtained for each of the color channels, and the materials are sorted according to increasing root mean square errors of the corresponding fitted Blinn-Phong model. The *bisquare weights* technique in *robustfit* algorithm in MATLAB was used for fitting the polynomial models. It is seen from the figure that in more than in half of the cases the corresponding root mean squared errors of the fitted polynomial models are less than that of the Blinn-Phong model. One reason for such a result might be that no weighting was used for the nonlinear Blinn-Phong model. Another reason is that the polynomial models might provide better representation than that of the Blinn-Phong model for the BRDF measurements of those materials.

A reference image of a sphere was obtained using Zernike polynomials and two created images of the same material that is orange peel based on the Blinn-Phong model and Polynomial model of degree p=4, 5, 6 and 7 are presented in Figure 4. The difference images were obtained by combining the corresponding difference images each of which were constructed by subtracting the test image from the created ones of each color channel. It is seen from the figure that the polynomial models generally provide better approximations to the reference image. However, the higher the degree of the polynomial the better the resulting approximation is observed. Similarly, images of the spheres were obtained for six different isotropic materials and the corresponding renderings are displayed in Figure 5. In all cases the polynomial model of degree 7 provided better approximations of the test image.

For a more detailed comparison, the peak signal-to-noise

Table 1. *PSNR values of images based on the Blinn -Phong model and polynomial model of degree 7 for different materials*

Material	PSNR Values					
	Blinn-Phong			Polynomial (p=7)		
	Red	Green	Blue	Red	Green	Blue
Orange P.	30,97	29,32	34,21	32,75	31,85	46,09
Terry-cloth	29,60	30,38	34,06	37,74	39,02	44,03
Velvet	32,50	40,05	61,87	40,71	50,04	∞
Cork	27,86	30,37	31,28	30,63	35,44	42,36
Rug_b	27,66	27,99	30,71	28,26	29,72	39,20
Loofa	31,40	33,23	36,93	36,00	39,15	44,91
Moss	27,78	27,84	37,59	32,59	33,22	45,89

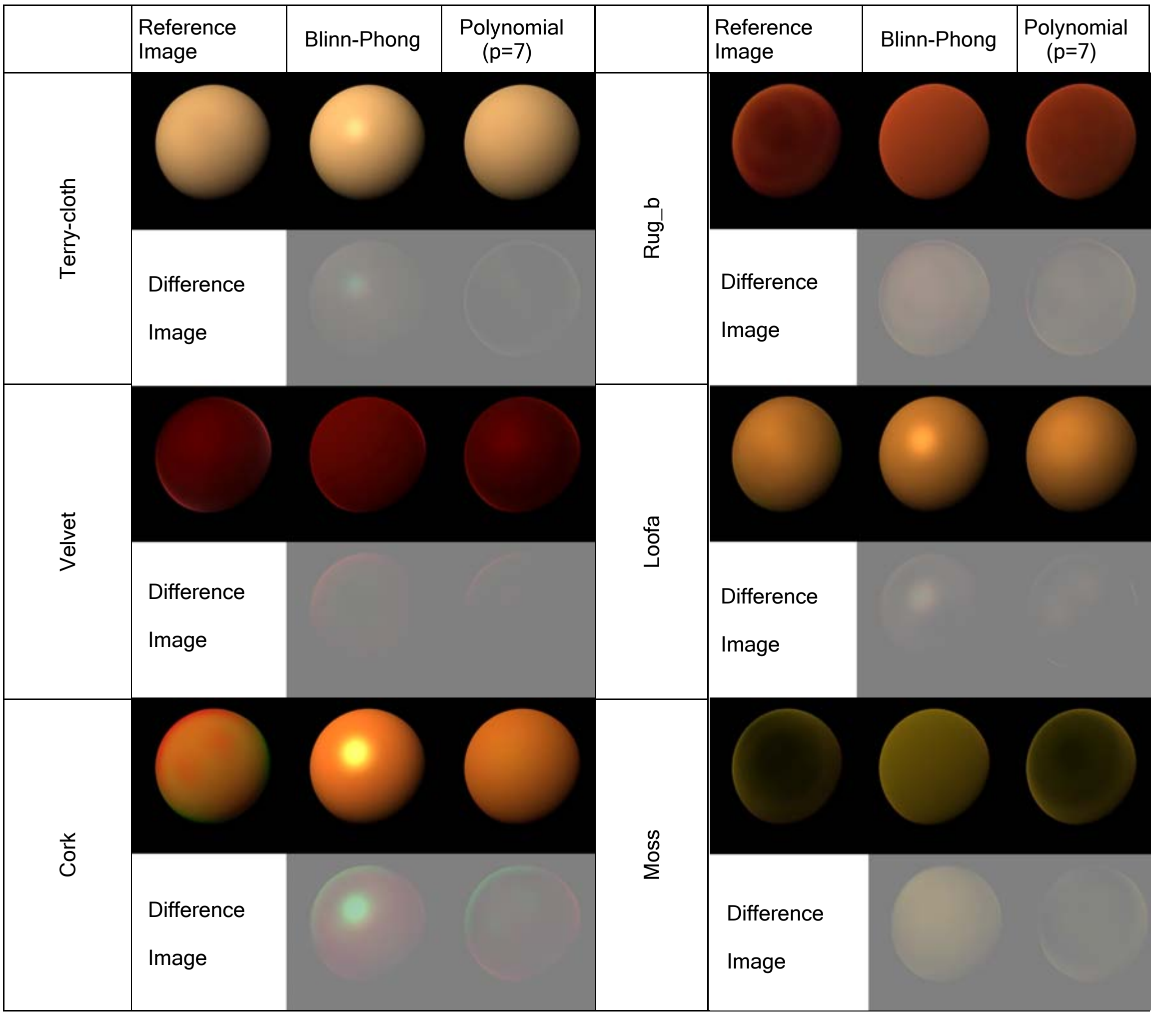

Figure 5. *Spheres rendered using Zernike polynomials, the Blinn-Phong and the polynomial models and the corresponding difference images for different materials.*

ratio (PSNR) values were computed and presented in Table 1 for each material used in Figures 4 and 5. Considering the six different materials, the results show that the PSNR values based on the polynomial model are uniformly higher than that of the Blinn-Phong model. We note that the blue channel images of the polynomial models are very close to those of the test images.

The Blinn-Phong model commonly is used to model specular highlights. In this study, the experimental results based on the CUReT data showed that the polynomial model provides better representation for BRDF than the Blinn-Phong model. To investigate the computational efficiency of the polynomial model, we generated artificial BRDF data using Blinn-Phong model in (6) with parameters $\mu = 0, \sigma = 1$ and $\gamma = 20$. Uniformly spaced $x = \mathbf{n} \cdot \mathbf{h}$ values in the interval 0<x<1 were used for each sample of size n=1000, 2000, 5000 and 10.000. Levenberg-Marquardt (*fsolve*) and *robustfit* algorithms in MATLAB were used to estimate the parameters of the Blinn-Phong model and the polynomial models respectively. The relative execution times of the polynomial models to the Blinn-Phong model were obtained on a Pentium 4 PC and presented in Table 2. It is seen from the table that the execution times of the polynomial model are shorter than that of the Blinn-Phong model for all cases. It is interesting to note that in an extreme case when the degree of the polynomial is p=10 and the sample size is n=10.000, the corresponding estimation procedure of the polynomial model is about 5 times faster than its competitor.

The Blinn-Phong model in (6) has a simple closed form. For a given set of parameters, the corresponding BRDF value can be computed easily. However, the computational cost of this model as compared with a linear model may be higher because of its exponential term. To investigate this issue we evaluated 100.000 BRDF values from (6) at a fixed point $x = x_0$ and measured the total execution time

Table 2. *Relative execution times for polynomial model of various degrees and sample sizes.*

	n=1000	n=2000	n=5000	n=10000
p=1	3.29	7.27	10.52	15.82
p=2	3.28	7.27	11.43	17.61
p=3	2.56	6.67	9.74	15.20
p=4	2.56	5.72	8.77	12.12
p=5	2.56	4.71	7.51	10.62
p=6	1.92	4.21	6.74	9.69
p=7	1.92	4.00	6.12	8.71
p=8	1.64	3.48	4.70	6.80
p=9	1.35	3.20	4.78	6.15
p=10	1.35	2.96	4.31	5.62

Table 3. *Relative execution times for evaluating models in (6) and (7) at point x=0.5.*

p=1	173.90	p=6	8.02
p=2	132.83	p=7	7.09
p=3	95.14	p=8	5.54
p=4	13.03	p=9	4.28
p=5	9.92	p=10	3.49

of this process. The polynomial model in (7) first was expressed by *Horner's rule* [CLRS01] as

$$f = \beta_0 + x_0(\beta_1 + x_0(\beta_2 + \cdots x_0(\beta_{p-1} + x_0\beta_p)\cdots) \quad (12)$$

and then evaluated in a similar fashion. The corresponding relative execution times are presented in Table 3 for polynomials of various degrees. The results show that the polynomial models considered in this comparison are always faster than the Blinn-Phong model. In a typical application when a polynomial model of degree 5 is used then the speed of evaluation of this model will be about 10 times faster than that of the Blinn-Phong model. These empirical results suggest that the polynomial models provide an important advantage for real-time applications.

4. Conclusions

In this paper we have approximated the Blinn-Phong model by a polynomial model of degree p. It is shown that the polynomial model provides satisfactory approximation of the Blinn-Phong model. However, the Blinn-Phong model is an empirical model and used to approximate the surface reflectance. In this sense, the proposed linear model can be considered independently as an approximation model of the surface reflectance. Using a BRDF test data, renderings based on the underlying models were obtained. For a number of materials, experimental results suggest that the polynomial models of degree greater than 5 perform well for modeling the surface reflectance.

The proposed model is linear in parameters and the corresponding parameters can be estimated easily by using standard regression procedures. The polynomial model is faster than the Blinn-Phong model both in the parameter estimation and in the model evaluation. Our final conclusion is that not only can the underlying model be used to approximate the Blinn-Phong model but also can be implemented independently to model the surface reflectance for certain class of applications. The quality of the simulated images obtained by the proposed approach can be improved, at the expense of storing additional parameter estimates.

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