Gathering for Free in Random Walk Radiosity

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Abstract. We present a simple technique that improves the efficiency of random walk algorithms for radiosity. Each generated random walk is used to simultaneously sample two distinct radiosity estimators. The first estimator is the commonly used shooting estimator, in which the radiosity due to self-emitted light at the origin of the random walk is recorded at each subsequently visited patch. With the second estimator, the radiosity due to self-emitted light at subsequent destinations is recorded at each visited patch. Closed formulae for the variance of the involved estimators allow to derive a cheap heuristic for combining the resulting radiosity estimates. Empirical results agree well with the heuristic prediction. A fair error reduction is obtained at a negligible additional cost.

Keywords: radiosity, Monte Carlo, random walk

1 Random walk estimators for radiosity

We first review the two random walk estimators for radiosity that will be combined in §2. A discussion of the technique and its results are presented in §3.

1.1 The shooting estimator

The distribution of light power P_i in a scene, discretised in patches i, can be obtained by solving the following system of linear equations[1]:

$$\begin{aligned} P_i &= \Phi_i + \sum_j P_j F_{ji} R_i \qquad (1) \\ &= \sum_{j_0} \Phi_{j_0} \delta_{j_0 i} + \sum_{j_0, j_1} \Phi_{j_0} F_{j_0 j_1} R_{j_1} \delta_{j_1 i} + \sum_{j_0, j_1, j_2} \Phi_{j_0} F_{j_0 j_1} R_{j_1} F_{j_1 j_2} R_{j_2} \delta_{j_2 i} + \cdots \end{aligned}$$

Now consider a random variable $\hat{b}^S_{is}$ taking values $\hat{b}^S_{is}(\mathbf{J})$ with probability $p^S(\mathbf{J})$ where

$$A_i \hat{b}^S_{is}(\mathbf{J}) = \delta_{s j_0} \frac{\Phi_s}{p_s} \cdot \sum_{k=1}^{n} R_{j_k} \delta_{j_k i} \qquad (2)$$

$$p^S(\mathbf{J}) = p_{j_0} \cdot F_{j_0 j_1} R_{j_1} \cdots F_{j_{n-2} j_{n-1}} R_{j_{n-1}} F_{j_{n-1} j_n} (1 - R_{j_n}). \qquad (3)$$

$\mathbf{J}$ denotes any sequence $j_0, \ldots, j_n, n \geq 1$ of patches. Such a random variable can be sampled by generating random walks $\mathbf{J}$ with origin selection probability p_{j_0}, transition

[1] The meaning of all symbols used in this paper is tabulated in table 1.

P_i	Total power emitted by patch i
Φ_i	Self-emitted power; $\Phi_T = \sum_s \Phi_s$
A_i	Area; $A_T = \sum_i A_i$ is the total surface area
R_i	Reflectivity; $R_{ave} = (1/A_T)\sum_i A_i R_i$ is the area-average reflectivity
B_i	Total radiosity emitted by i;
E_i	Self-emitted radiosity; $E_{ave} = \Phi_T/A_T$: average self-emitted radiosity
b_i	Non-self-emitted radiosity $b_i = B_i - E_i$
b_{is}	Radiosity on i due to self-emitted radiosity on source s; $b_i = \sum_s b_{is}$
$\hat{b}_{is}$	An estimator for b_{is}; $\hat{b}_i = \sum_s \hat{b}_{is}$; $\hat{b}_i^S$: shooting estimator, $\hat{b}_i^G$: gathering estimator.
$E[\hat{b}_{is}]$	Expectation of the estimator $\hat{b}_{is}$
$V[\hat{b}_{is}]$	Variance of the estimator $\hat{b}_{is}$
$\mathbf{J}$	A sequence of patches $j_0, \ldots, j_n$ (used to denote a random walk)
$\hat{b}_{is}(\mathbf{J})$	Contribution of the random walk $\mathbf{J}$ to the estimator $\hat{b}_{is}$
p_i	Probability of starting a random walk on i
$p(\mathbf{J})$	Probability of generating the random walk $\mathbf{J}$
δ_{ij}	Kronecker's delta: 1 if $i = j$ and 0 if $i \neq j$
ξ_i	Incident power received back at i due to emission of one unit of power by i

Table 1. Symbols used in this paper. Symbols like i, s, $j_0, \ldots, j_n$ denote a patch.

probabilities equal to the form factor F_{kl} from patch k to patch l and survival probabilities equal to the reflectivity R_k on each patch k. The transitions can be simulated using local lines, as in [8], or global lines [7]. If such a random walk originates at s, a contribution of $R_i\Phi_s/A_i p_s$ is recorded each time the patch i is visited. No contribution is recorded at the origin j_0 itself of a random walk however.

It can be shown that the expectation $E[\hat{b}_{is}^S] = \sum_{\mathbf{J}} \hat{b}_{is}^S(\mathbf{J})p^S(\mathbf{J})$ of this random variable equals the radiosity b_{is} on i due to self-emitted radiosity on the source s. The sums $\hat{b}_i^S(\mathbf{J}) = \sum_s \hat{b}_{is}^S(\mathbf{J})$ over all sources s in the scene yield an estimator $\hat{b}_i^S$ for the total non-self-emitted radiosity b_i. It can be shown that the variance is given by [4, 5]:

$$V[\hat{b}_i^S] = \frac{R_i(1+2R_i\xi_i)}{A_i}\sum_s \frac{\Phi_s}{p_s} b_{is} - b_i^2. \tag{4}$$

This random walk estimator leads to a "discretised" version of the particle tracing algorithm [2]. In an implementation, incident particles are "warped" to a uniformly chosen other position on each hit patch. It is closely related to various other Monte Carlo radiosity algorithms.

1.2 The gathering estimator

A similar random walk estimator can be derived from the radiosity equations:

$$\begin{aligned} B_i &= E_i + \sum_j R_i F_{ij} B_j \qquad (5) \\ &= \sum_{j_0} \delta_{ij_0} E_{j_0} + \sum_{j_0,j_1} \delta_{ij_0} R_{j_0} F_{j_0 j_1} E_{j_1} + \sum_{j_0,j_1,j_2} \delta_{ij_0} R_{j_0} F_{j_0 j_1} R_{j_1} F_{j_1 j_2} E_{j_2} + \ldots \end{aligned}$$

Consider the random variable $\hat{b}_{is}^G$ taking values $\hat{b}_{is}^G(\mathbf{J})$ with probability $p^G(\mathbf{J})$ where

$$\hat{b}_{is}^G(\mathbf{J}) = \delta_{ij_0} \frac{R_i}{p_i} \cdot \sum_{k=1}^{n} E_{j_k} \delta_{j_k s} \tag{6}$$

$$p^G(\mathbf{J}) = p_{j_0} \cdot F_{j_0 j_1} R_{j_1} \cdots F_{j_{n-2} j_{n-1}} R_{j_{n-1}} F_{j_{n-1} j_n} (1 - R_{j_n}). \tag{7}$$

Again, $\mathbf{J}$ denotes a sequence of patches $j_0, \ldots, j_n, n \geq 1$. The random variable $\hat{b}_{is}^G$ can be sampled by generating random walks as for sampling $\hat{b}_{is}^S$. This time however, only walks originating from i instead of s will contribute. If originating at i, a contribution $R_i E_s$ is recorded every time the light source s is visited during the random walk. Also in this case, only visits $j_k = s$ for $k \geq 1$ count.

It can be shown that the expectation $E[\hat{b}_{is}^G]$ equals b_{is} as well. The sums $\hat{b}_i^G(\mathbf{J}) = \sum_s \hat{b}_{is}^G(\mathbf{J})$ over all sources s yield an estimator $\hat{b}_i^G$ for b_i with variance [4]:

$$V[\hat{b}_i^G] = \frac{R_i}{p_i} \sum_s (E_s + 2b_s) b_{is} - {b_i}^2. \tag{8}$$

This random walk estimator leads to an algorithm that is similar to ray-tracing. No next event estimators (shadow rays) are traced however and incident particles are "warped" to a uniformly chosen new point on each hit patch. Instead of using next-event estimators, direct illumination can be used as a source light distribution rather than self-emitted illumination. Direct illumination can be computed first using a depth-one shooting pass [6]. A more advanced such "smoothing" pass is proposed in [1].

2 The new algorithm: gathering for free

2.1 Simultaneous shooting and gathering

Consider first a fixed pair of patches s and i. Consider any random walk $\mathbf{J} = j_0, \ldots, j_n$, $n \geq 1, j_0 = s, j_n = i$ originating at s and being absorbed on i, but furthermore generated as described above. The probabilities $p^S(\mathbf{J})$ (3) and $p^G(\mathbf{J})$ (7) are identical for each such a random walk. They can therefore be used to sample both estimators $\hat{b}_{is}^S$ and $\hat{b}_{si}^G$ simultaneously: to "gather" an amount of radiosity $\hat{b}_{si}^G(\mathbf{J})$ (6) at s from i[2] while "shooting" an amount of radiosity $\hat{b}_{is}^S(\mathbf{J})$ (2) from s to i.

Each random walk can however be used to obtain gathering or shooting contributions to the total non-self-emitted radiosity b_{j_k} at *every* visited patch j_k (see figure 1):

- Shooting: the radiosity due to all sources s is estimated by generating random walks from each source s with probability $p_s = \Phi_s / \Phi_T$. A contribution $R_{j_k} \Phi_T / A_{j_k}$ is recorded at every visited patch $j_k, k \geq 1$ (no contribution at the origin);
- Gathering: the radiosity at s is estimated more efficiently by recording a gathering contribution at s for every visited patch $j_k, k \geq 1$. Moreover, since each sub-path $j_k, \ldots, j_n, k \geq 1$ is an independent path for every visited patch j_k [3], it is allowed to accumulate a gathering contribution at each $j_k, k < n$ for each subsequently visited patch $j_l, l > k$[3]. In short, a gathering contribution of $R_{j_k}(E_{j_{k+1}} + \cdots + E_{j_n})$ shall be recorded at each $j_k, k = 0, \ldots, n-1$.

In an implementation, shooting and gathering contributions shall be accumulated separately on each patch i. Eventually, the shooting contributions at i shall be divided by the total number N of random walks. The gathering contributions at i shall be divided by the number of gathering contributions N_i^G at i. After adding self-emitted radiosity E_i, two independent estimates B_i^S and B_i^G for the radiosity on each path i are obtained.

[2] Note the switch of indices compared to (6).

[3] It is possible to use the sub-paths for shooting as well, but this results in increased variance.

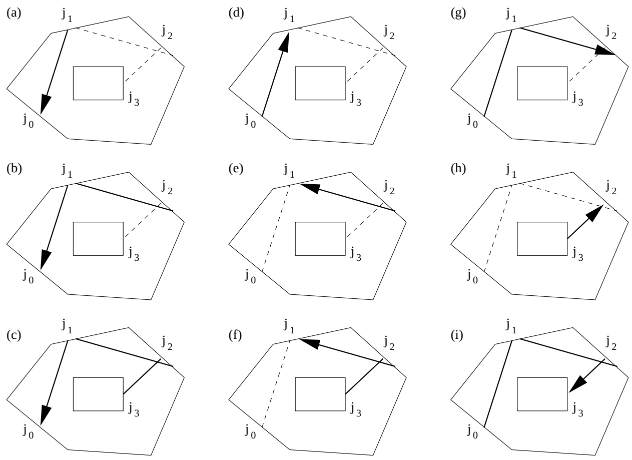

Figure 1. Contributions of a random walk j_0, j_1, j_2, j_3: (a,b,c) gathering at j_0; (d) shooting at j_1; (e,f) gathering at j_1; (g) shooting at j_2; (h) gathering at j_2; (i) shooting at j_3.

The gathering estimates B_i^G are obtained at negligible extra computation cost. Storage requirements are however slightly higher due to the need to store the count of gathering contributions N_i^G per patch as well as two radiosity estimates instead of one.

2.2 Combination of the radiosity estimates

The optimal combination $\alpha_i B_i^S + \beta_i B_i^G$ is obtained by chosing $\alpha_i + \beta_i = 1$ with each coefficient inverse proportional to the variance of the corresponding estimators:

$$\frac{\beta_i}{\alpha_i} = \frac{V[\hat{b}_i^S]/N}{V[\hat{b}_i^G]/N_i^G}. \tag{9}$$

Closed formulae for the variances $V[\hat{b}_i^S]$ and $V[\hat{b}_i^G]$ were given above. Unfortunately, these formulae require very detailed knowledge of the radiosity solution, which is not available in practice. Near-optimal weights can be obtained however by using approximations for the variances in (9).

A very simple but reasonably good heuristic for determining the weights is obtained by introducing the following assumptions:

- The origin selection probability for the random walks is $p_s = \Phi_s/\Phi_T$ in (4), and only those random walks that yield a gathering contribution at patch i are counted in N_i^G ($p_i = 1$ in (8));
- $R_i\xi_i$, the fraction of power received back at i due to own emission, is small in (4);
- $\sum_s (E_s + 2b_s) b_{is} \approx (\sum_s A_s(E_s + 2b_s)/A_T) \cdot \sum_s b_{is}$ in (8);
- After a smoothing pass, almost every patch i can be considered a source [6, 1], so that $\sum_s A_s b_s \approx \sum_i A_i b_i \approx \frac{R_{ave}\Phi_T}{1-R_{ave}}$;

With these assumptions, the following approximation for (9) is obtained:

$$\frac{\beta_i}{\alpha_i} \approx \frac{N_i^G A_T}{N A_i} \cdot k \approx \frac{B_i}{E_{ave}} \cdot k \quad \text{with optimal } k = \frac{1 - R_{ave}}{1 + R_{ave}}. \tag{10}$$

The second alternative follows from $E[N_i^G] = NP_i/\Phi_T$, which is easy to prove, and suggests the use of a (a-posteriori) radiosity estimate, e.g. $B_i^S \approx B_i$, instead of N_i^G.

3 Results and discussion

The combination is asymptotically unbiased. Since $E[\alpha_i B_i^S + \beta_i B_i^G] = E[B_i^S + \beta_i (B_i^G - B_i^S)]$ and $E[B_i^S] = B_i$, the bias is given by $E[\beta_i (B_i^G - B_i^S)]$. Since $\beta_i \leq 1$, the bias is bounded by $E[|B_i^G - B_i^S|] = \sqrt{\frac{2}{\pi}(V[\hat{b}_i^S]/N + V[\hat{b}_i^G]/N_i^G)}$ for sufficiently large N and N_i^G so that the central limit theorem applies.

The heuristic weights are reasonable. Figure 2 show that in three tested scenes with average reflectivity 0.2, 0.45, and 0.8 the choice of k in (10) is near to the optimal choice indeed. The assumptions in §2.2 are satisfied well in these scenes.

Fair error reduction at nearly no additional cost. In the tested scenes, a mean square error (MSE) reduction of 9.8%, 19%, and 49% respectively was observed. Resulting images for the scene with average reflectivity 0.8 are shown in figure 3. A reduction of the MSE by 49% does not translate in dramatic improvements in visual appearance. The reduction of the error is however obtained at nearly no additional computation cost. With shooting only, 49% more random walks would be needed in order to achieve a given error level.

Related work. In [9], heuristics are presented for combining an a-priori known number of samples of a single integrand drawn from several probability distributions. In our case, we deal with samples of two distinct sums drawn from a single probability distribution. It is possible to reformulate the problem so that the heuristics in [9] can be applied when gathering to/shooting from only the origin of the paths. The heuristics in [9] cannot be used for combining shooting and gathering over all sub-paths: this would require that the probability that a random walk visits any patch is known in advance. These probabilities are proportional to the flux P_i of the patches, which is the result to be computed. For our combination heuristic, a a-posteriori radiosity estimate is sufficient.

4 Conclusion

The combination of a shooting random walk estimator with a corresponding gathering estimator which can be sampled at negligible additional cost can yield fair a error reduction. The technique was presented for the most commonly used shooting estimator, but can be used equally well with other random walk radiosity estimators as well as with similar estimators for general environments. The main area for improvement is in the development of more elaborate heuristics to combine the estimates by better approximation of the variance formulae.

References

1. F. Castro, R. Martínez, and M. Sbert. Quasi monte carlo and extended first shot improvement to the multi-path method for radiosity. In *Proceedings of SCCG'98, Budmerice, Slovakia*, April 1998.
2. S. Pattanaik and S. Mudur. Computation of global illumination by Monte Carlo simulation of the particle model of light. *Third Eurographics Workshop on Rendering*, pages 71–83, May 1992.
3. R. Y. Rubinstein. *Simulation and the Monte Carlo method*. J. Wiley and sons, 1981.

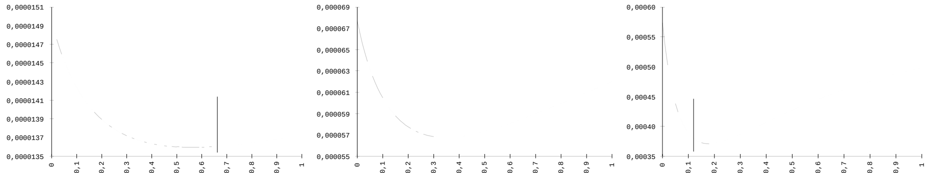

Figure 2. Mean square error (vertical axis) against combining factor *k* in (10) for a scene with average reflectance 0.2 (left), 0.45 (middle) and 0.8 (right). The vertical line indicates the proposed value for *k*. The scene with reflectance 0.8 is shown in figure 3.

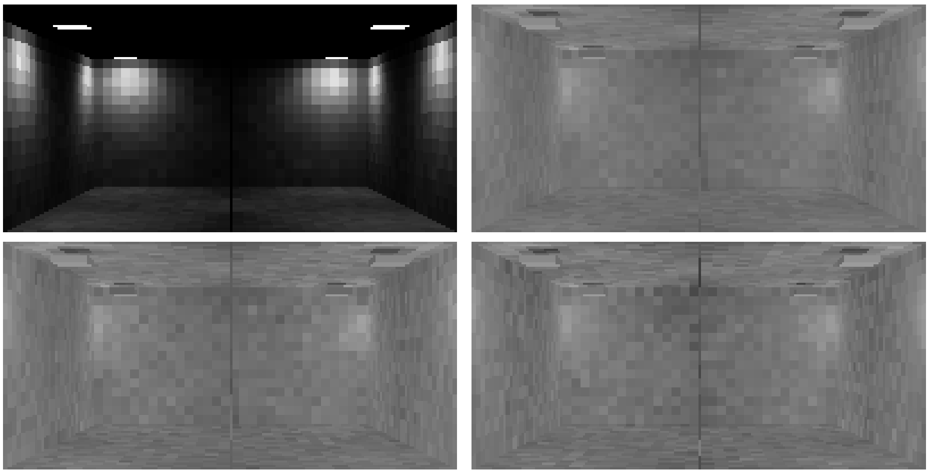

Figure 3. Top left: direct illumination used as the source distribution; Top right: Indirect illumination by combining shooting (bottom left) and gathering (bottom right).

4. M. Sbert. Error and complexity of random walk Monte Carlo radiosity. *IEEE Transactions on Visualization and Computer Graphics*, 3(1):23–38, March 1997.
5. M. Sbert. Optimal source selection in shooting random walk monte carlo radiosity. *Computer Graphics Forum*, 16(3):301–308, August 1997. Proc. Eurographics '97.
6. M. Sbert. *The use of global random directions to compute radiosity. Global Monte Carlo methods*. PhD thesis, Universitat Politècnica de Catalunya, Barcelona, March 1997. Available from `http://ima.udg.es/~ mateu`.
7. M. Sbert, X. Pueyo, L. Neumann, and W. Purgathofer. Global multipath monte carlo algorithms for radiosity. *The Visual Computer*, 12(2):47–61, 1996.
8. P. Shirley. Radiosity via ray tracing. In James Arvo, editor, *Graphics Gems II*, pages 306–310. Academic Press, San Diego, 1991.
9. E. Veach and L. Guibas. Optimally combining sampling techniques for Monte Carlo rendering. In *Computer Graphics (ACM SIGGRAPH'95 conference proceedings)*, 1995.

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