Computational Aesthetics in Graphics, Visualization, and Imaging (2012)
D. Cunningham and D. House (Editors)

Surface Covering Curves

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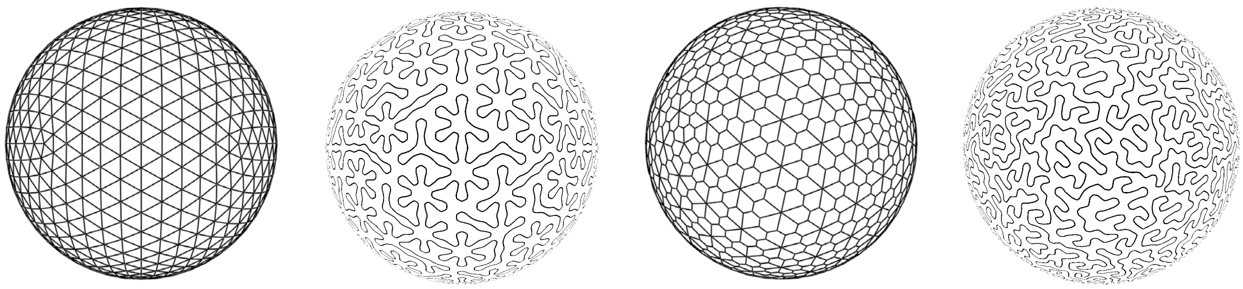


(a) Geodesic dome | (b) Curve constructed from the mesh in (a) | (c) Another spherical mesh | (d)Curve constructed from the mesh in (c)

Figure 1: *Sphere covering examples of surface filling curves: Spherical mesh surfaces in (a) and (b) are converted to a closed 3D curves which follow the shapes of the original spheres.*

Abstract

In this work, we present the concept of surface covering curves that can be used to construct wire sculptures or surface textures. We show that any mesh surface can be converted to a single closed 3D curve that follows the shape of the mesh surface. We have developed two methods to construct corresponding 3D ribbons and yarns from the mesh structure and the connectivity of the curve. The first method constructs equal thickness ribbons (or equal diameter yarns). The second method creates ribbons with changing thickness (or yarns with changing diameter) that can densely cover the mesh surface.
Since each iteration of any subdivision scheme results in a denser mesh, the procedure outlined above can be used to obtain a denser and denser curve. These curves can densely cover a mesh surface in limit. Therefore, this approach along with a subdivision scheme provides visual results that are similar to space filling curves that are created by fractal algorithms. Unlike space filling curves which fills a square or a cube, our curves cover a surface, and henceforth, we called them "surface covering curves". Space covering curves also resemble TSP (traveling salesmen problem) art and Truchet-like curves that are embedded on surfaces.

1. Introduction and Motivation

In this work, we provide a simple approach to create aesthetic curves in 3-space. Our approach is based on converting mesh surfaces to closed 3D curves that follow the shapes of the given mesh surfaces. Our work is based on Gabriel Taubin's work on constructing Hamiltonian triangle strips on quadrilateral meshes [Tau03].

In graph theory, a Hamiltonian path is a path in an undirected graph that visits each vertex exactly once. A Hamiltonian cycle (or Hamiltonian circuit) is a Hamiltonian path that is a cycle. For any graph Hamiltonian cycle may not

DOI: 10.2312/COMPAESTH/COMPAESTH12/107-114

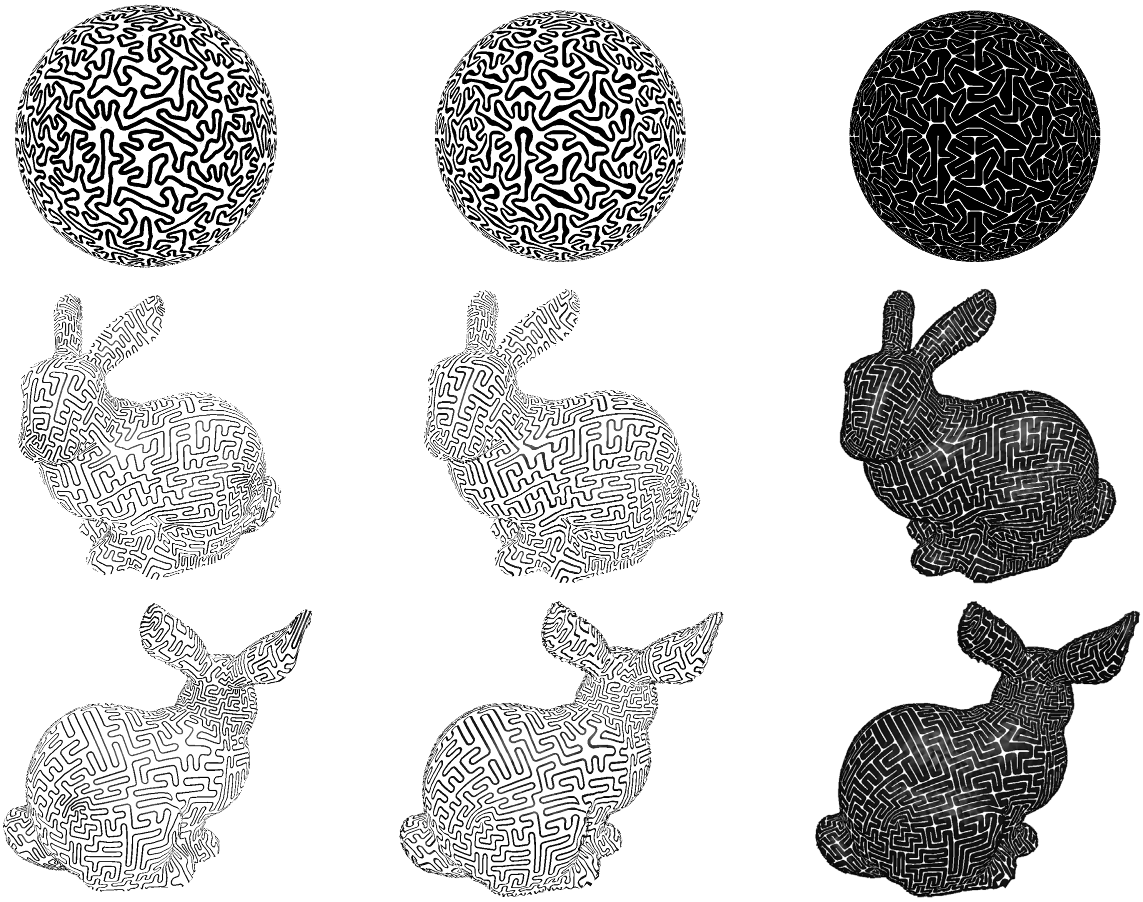
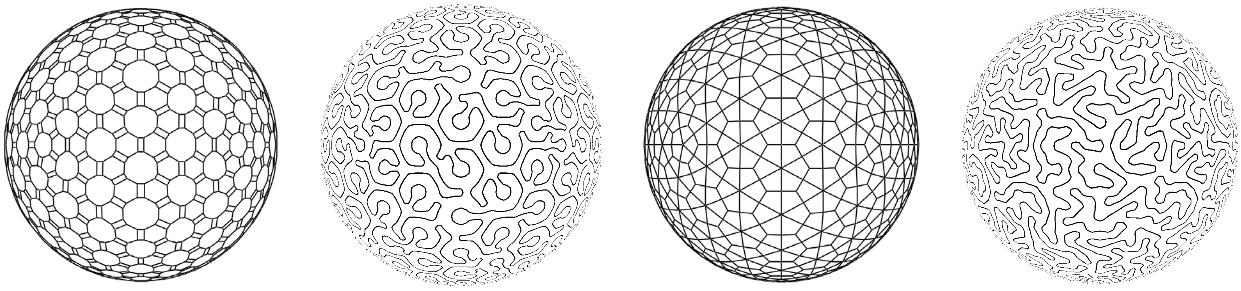


Figure 2: *Dense covering of surfaces using ribbons with changing diameter. The parts of the curves that are occluded by original surfaces are not drawn for cleaner images.*

necessarily exist. Hamiltonian triangle strips are defined in dual of triangular meshes. Taubin show that it is always possible to construct a triangular mesh from any given quadrilateral mesh such that the dual of the triangular mesh has an Hamiltonian cycle. Moreover, he presented simple linear time and space constructive algorithms to construct these triangle strips, His algorithms are based on splitting each quadrilateral face along one of its two diagonals and linking the resulting triangles along the original mesh edges. With these algorithms every connected manifold quadrilateral mesh without boundary can be represented as a single Hamiltonian generalized triangle strip cycle. Based on these algorithms to construct a closed curve is also straightforward. One can simply connect centers of triangles in the triangle strip to obtain a control polygon in 3D. Resulting control polygon can be turned into a smooth curve using a parametric curve such as B-Spline as shown in Figure 1 and 3.

Since the shape of any given surface can be approximated by a wide variety of meshes, designers of these curves have significantly large number of aesthetic possibilities. Two examples that show the effect of underlying mesh are shown in Figure 1. Moreover, even for a given mesh there are exponentially many ways to form these curves since with probability 1, there are 2^{F-1} Hamiltonian cycles for any given M where F is the number of faces of mesh M (see [XACG10] for a related problem). This property provides additional aesthetic possibilities since designers can have additional control over the shapes of the curves. We prefer curves with multiple points of inflection (wavy curves) since they resemble space filling curves [Man82] or TSP (traveling salesmen problem) art [KB05] embedded on surfaces.

Since curves are on surfaces, it is also easy to convert



(a) A spherical mesh (b) Curve constructed from the mesh in (a) (c) Another spherical mesh (d)Curve constructed from the mesh in (c)

Figure 3: *More sphere filling curves: Spherical mesh surfaces in (a) and (b) are converted to a closed 3D curves which follow the shapes of the original spheres. Back-faces in meshes and back-face parts of the curves are not drawn for cleaner images.*

them to 3D structures that can be shaded, rendered and even eventually 3D printed. We have developed two methods to construct corresponding 3D ribbons and yarns from given curves as extruded lines and polygons along the curves. The first method, called constant-diameter, simply turns the curves constant thickness ribbons or equal diameter yarns. The second method, which we call variable-diameter, creates ribbons with varying thicknesses (or yarns with changing diameters) that can densely cover the mesh surface. We have developed a system that converts polygonal meshes to surface filling curves, ribbons and yarns. All the images in this paper are direct screen captures from the system. They were created in real-time.

Figure 2 shows an example obtained by using constant and variable diameter methods. Our variable diameter method guarantees that the sizes are relative to the underlying triangles. Therefore, the actual widths of ribbons are different in different parts of the mesh. Fig 8 shows visual effects of constant vs. variable and ribbon vs. tread for the same spherical mesh.

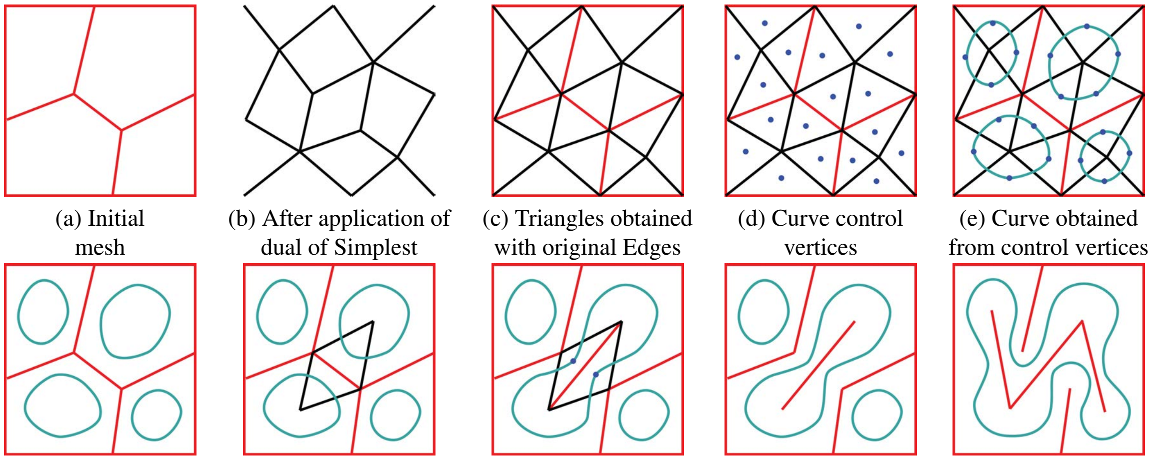
2. Previous Work

Space-filling curves, which are discovered by Giuseppe Peano [Pea90] by his construction of a continuous mapping from the unit interval onto the unit square, are a mapping from one-dimensional space to a multi-dimensional space. A space filling curve is like a thread that passes through every cell element in the multi-dimensional space so that every cell is visited at least once. Space filling curves have became very well-known among mathematician/artists after Benoit Mandelbrot's seminal work on Fractal Geometry [Man82]. In his book, he categorized space filling curves as fractals since they can be constructed using a replacement algorithm starting from a simple shape. Mathematician and artist Douglas McKenna [McK78], who also created many images in Mandelbrot's Fractal Geometry of Nature, also discovered one space-filling curve. McKenna [McK78] later enumerated over 20 million new space-filling recursive designs. Most of existing examples of space filling curves are in 2D. A remarkable exception is Carlo Sequin's stainless steel and bronze sculpture called Hilbert Cube 512 [Séq06]. This sculpture is a closed (thickened) curve that fills the volume of a cube.

Robert Bosch and Adrianne Herman invented another related artwork. called TSP (traveling salesmen problem) art [BH04, BH04]. In TSP art a set of points that we can think of as cities. A traveling salesman who reside one of the cities want to visit each of the other cities exactly once and then return home. The salesman would like to visit the cities in an order that will minimize the total length of his tour. Bosch and Herman noticed that for interestingly placed city locations, the piecewise curve that show salesman's itinerary looks artistic. To create original artwork, they used points on a grid. This method was simple but required many dots to produce a decent picture since the dots tended to clump together. Bosch and Kaplan [KB05] used weighted Voronoi stippling to control positions of the cities. By distributing cities with a density that locally approximates the darkness of a source image, and passing the cities to a program that finds a TSP tour, they have produced TSP-art that resembles the source image.

Another related work is Truchet tiles, which was originally introduced by Sebastien Truchet as all possible patterns formed by tilings of right triangles oriented at the four corners of a square [Tru04]. The work related to ours is introduced by Clifford A. Pickover [Pic89] as a single tile consisting of two circular arcs of radius equal to half the tile edge length centered at opposed corners . The two possible orientations of this tile, and tiling the plane using tiles with random orientations gives visually interesting curves called Truchet curves [Bro07]. Truchet curves separates the plane into two regions and therefore it is used to create planar artworks [Bro08].

In this paper, we show that it is always possible to ob-



(a) Initial mesh (b) After application of dual of Simplest (c) Triangles obtained with original Edges (d) Curve control vertices (e) Curve obtained from control vertices

(f) Resulting curves (g) An old edge selected (h) Old edge flipped New control vertices (i) Combined curve (j) Final result

Figure 4: *Visual presentation of the Hamiltonian cycle construction and curve generation algorithm.*

tain a single closed curve that covers a surface similar to TSP art and space filling curves. In terms of visual aesthetics, our curves resemble the most to Truchet curves. In fact, if our method is applied a planar grid, the result will be a single Truchet curve. Although our curves covers space similar to space filling curves, they are not strictly self-similar, i.e. fractals. However, our results exhibit similarities that are visible in our examples. These similarities are just result of structure of underlying mesh and initial choices.

3. Overall Framework

Our approach can be considered a 2-step process: (1) identify a Hamiltonian cycle that connects vertices of dual of a given mesh, (2) use the 3D positions of vertices (i.e. face-centers of the original mesh) as control vertices of a smooth curve. The resulting curve is guaranteed to be closed and follow the overall shape of the surface. We point out that it is NP-hard to find an Hamiltonian cycle for even cubic 3-connected planar graphs [GJT76]. It is also known that existing exponential-time algorithms for Hamiltonian cycles are not sufficient to find single strips for triangular meshes more than 100 triangles [Epp03]. Fortunately, many researchers observed that the hardness of finding Hamiltonian cycles can be simplified by minor variations of the problem statement. For example, by adding a few new triangles, it is possible to significantly simplify the Hamiltonian cycle problem without changing the input geometry and visual quality [GE04].

Taubin showed that from a quadrilateral manifold mesh, it is possible to construct a triangular mesh with an associated Hamiltonian cycle in linear time. The construction algorithm simply splits each quadrilateral into triangles and flip edges until the triangles are ordered into a single strip [Tau03]. The process is especially simple if the vertices of initial quadrilateral mesh is 2-colorable. Such 2-colorable quadrilateral meshes can be obtained by some subdivision schemes such as Catmull-Clark [CC78] and dual of Simplest [PR97] subdivisions.

4. Methodology

Our algorithm consists of two stages: (1) Hamiltonian cycle construction for curve generation, (2) Conversion of curves to varying-diameter ribbons or yarns. For curve generation, we use a variation of Taubin's method for fast computations Hamiltonian cycles. To convert curves to varying-diameter ribbons and yarns, we use a variation of projection method, which is presented to obtain weaving structures [ACXG09].

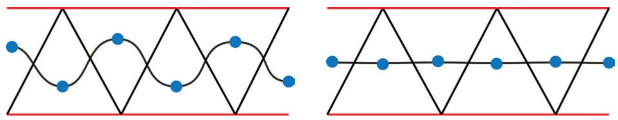


Figure 5: *The effect of weighted average that favors one vertex of triangle.*

4.1. Curve Generation

The curve generation consists of 6 steps:

- *Initial Mesh:* Initial mesh can be any manifold mesh surface of arbitrary topology. Although, we do not have any restriction, it is better to have only convex faces for aesthetic results. See Figure 4(a) where the original edges are drawn in red color.

- *Dual of Simplest:* To get initial quadrilateral, mesh we apply the subdivision scheme, which is the dual of Simplest subdivision. This subdivision can be obtained as two operations: (1) Simplest subdivision, the (2) dual operation. After this subdivision, each original edge of the initial mesh turns to a quadrilateral (see Figure 4(b) where newly created edges are drawn in black color.)
- *Initial Triangulation:* We insert old edges to turn all quadrilaterals to triangles as shown in Figure 4(c). Now, every triangle have two black edges and one red edge.
- *Control Vertex Position Computation:* For each triangle, we compute a center point as a weighted average of its vertex positions. Let $p_{0,0} = p_{0,1}$ denote the position of the vertex that is in the intersection of two black edges, and $p_{1,0}$ and $p_{1,1}$ denote the positions of the other two vertices. In other words, we treat the triangle as a quadrilateral of which two consecutive vertices share the same position. Based on this idea, control vertex position is computed as follows:

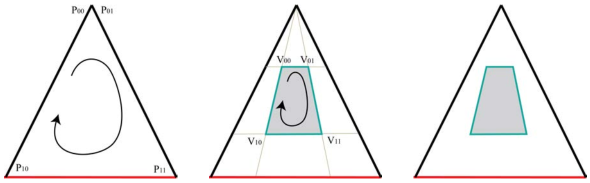
$$p_{cv} = \frac{p_{0,0} + p_{0,1} + p_{1,1} + p_{0,1}}{4}$$

Since $p_{0,0} = p_{0,1}$, this computation is a weighted average of vertex positions of triangle as follows:

$$p_{cv} = \frac{2p_{0,0} + p_{1,0} + p_{1,1}}{4}$$

These points, which serve as control vertices of surface filling curve, are shown in Figure 4(d).
Remark 1: Using weighted average helps to avoid higher frequency components when the connections do not supposed to create high frequencies as visually shown in Figure 5. Weighted average moves the control vertex to the middle of triangle height along the curve direction.
- *Initial Curves:* We construct a control polygon by connecting center points. For this purpose, if two vertices share a black edge, which is created by dual of Simplest subdivision, we connect these two vertices with an edge. After this operation, each original face is replaced by a closed curve as shown in Figures 4(e) and 4(f).

- *Combining Curves:* We, now, randomly choose an original edge (i.e. red edges in the Figures 4) and flip it if it is between two separate curve (see Figure 4(g)). After the flip, we recalculate triangle centers again and reconstruct the curve. As shown in Figure 4(h) this operation connect the two curves into one. We continue this operation until we obtain one curve as shown in Figure 4(j).
Remark 1: Twisted edges form a spanning tree for the dual of the initial mesh. In other words, this spanning tree connects all faces of the initial mesh as it can be seen in Figure 4(j).
Remark 2: After the flip operation, each triangle still have two black edges and one red edge.



(a) (b) (c)

Figure 6: *Computation of the trapezoid inside of the triangle.*

4.2. Geometry Conversion

By the curve generation algorithm, we create a single control polygon that passes from control points in 3 space. To obtain a smooth curve, the control polygon can be approximated or interpolated using a parametric curve such as B-Spline or Catmull-Rom curve [BBB87]. Since these curves do not have a solid shape, it is better to convert these control polygons to 3D structures such as ribbons (extruded lines alon the curves) or yarns (extruded polygons along the curves). For aesthetic purposes, the resulting 3D structures must look smooth and must not self-intersect. We have developed two methods for converting curves to smooth ribbons and yarns. We call these constant and variable diameter methods.

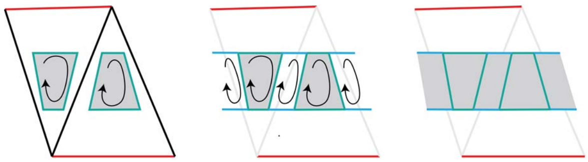

(a) (b) (c)

Figure 7: *Connecting the trapezoids with quadrilaterals.*

Constant diameter method is simply a line or a polygon extruded along the curve. On of our goals is to create dense covering in such a way that the ribbons or the yarns cover the surface without leaving large gaps. Constant diameter method provides nice thin and smooth curves but cannot densely cover the surface without self-intersection. We have introduced variable diameter method to provide dense covering. Our variable diameter method is related to projection method introduced for creating weaving cycles [ACXG09].

The variable diameter method consists of three steps.

- *Create a trapezoid inside of each triangle using two size parameters:* As we have discussed earlier, each triangle has one red (original or twisted original) and two black edges. Let $p_{0,0} = p_{0,1}$ denote the position of the vertex that is in the intersection of two black edges, and $p_{1,0}$ and $p_{1,1}$ denote the positions of the other two vertices (see Figure 6(a)). In other words, we again treat the triangle as a quadrilateral of which two consecutive vertices share the same position. Based on this idea, it is easy to compute the

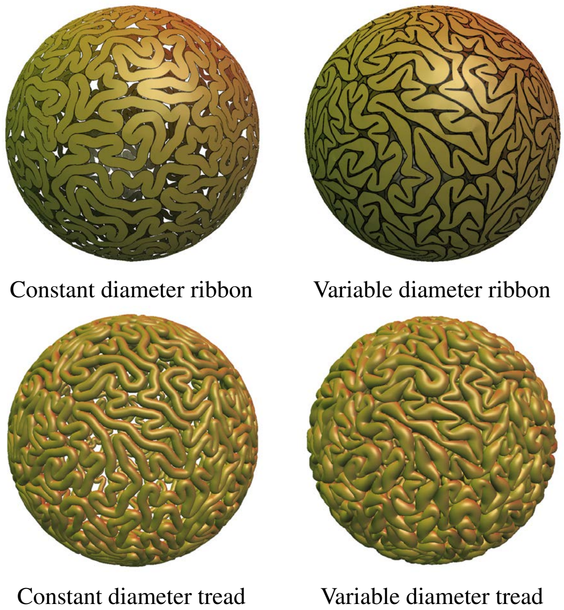


Figure 8: *An example that shows the visual effects of constant vs. variable and ribbon vs. tread for the same mesh. Back-face parts of the ribbons/yarns are also shown.*

positions of corners of a trapezoid that is drawn inside of this triangle simply using bilinear equation. Let $v_{0,0}$, $v_{0,1}$, $v_{1,0}$, $v_{1,1}$ denote the positions of four corners of quadrilateral drawn inside of the triangle (see Figure 6(b)). Then

$$v_{m,n} = \sum_{i=0}^{1}\sum_{j=0}^{1} \frac{(1-(-1)^i s)(1-(-1)^j t)}{4} p_{i+m,j+n}$$

where t and s are two parameters between 0 and 1; and the summations $i+m$ and $j+n$ are in modulo 2. *Remark 1:* For $s = t = 0$ the bilinear equation gives weighted average we have already used for computing control vertices.
Remark 2: Rotation order of vertices for trapezoid is important since it must give the same normal direction as the triangle. The bilinear equation guarantees the consistency of rotation order.
Remark 3: To create yarns, we simply extrude each trapezoid inside triangle in normal direction to obtain a trapezoidal prism.

- *Connect trapezoids in two consecutive triangles using a quadrilateral connector:* This operation simply insert two edges to form connectors as shown in Figure 7(b). In figure, the newly inserted edges are colored in darker blue. This operation turns initial triangular strip into quadrilateral strip. *Remark 1:* Rotation order of vertices for connectors must also be consistent with two neighboring trapezoids. Since we start with a manifold mesh, the original triangles always have consistent rotation order to start with.
Remark 2: This operation also guarantees that if a part of the original triangle strips forms a parallelogram, the same part of the resulting quadrilateral strips also form a parallelogram. In other words, if original data is not wavy, the resulting ribbon is guaranteed not to be wavy.
Remark 3: To create yarns, we connect trapezoidal prisms using hexahedral connectors, which are 3D versions of connectors in ribbon case. As a result, we obtain a generalized toroidal shape.
- *Smooth the quadrilateral strip using a subdivision scheme:* For smoothing resulting quadrilateral strips we use Catmull-Clark subdivision, which gives B-spline surfaces for regular structures such as quadrilateral strips [CC78]. As a result, variable diameter method provides almost the same shapes for thin ribbons. However, even for thicker ribbon it does not self-intersect until it covers the underlying surface with almost no gap. *Remark 1:* To smooth yarns, we simply smooth generalized toroidal shape, which can again be smoothed using Catmull-Clark subdivision. The result is the same as B-spline surface since toroidal shape consists of only quadrilaterals and 4-valent vertices [CC78].

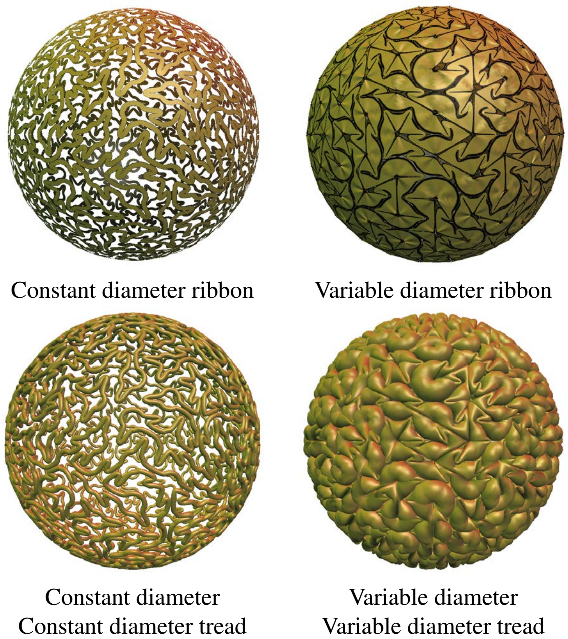
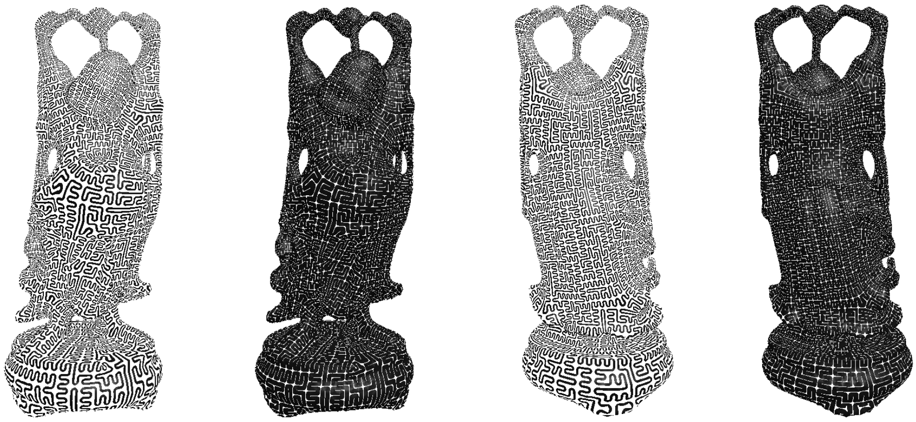


Figure 9: *An example that shows the visual effects of constant vs. variable and ribbon vs. tread for the same mesh. Back-face parts of the ribbons/yarns are also shown.*

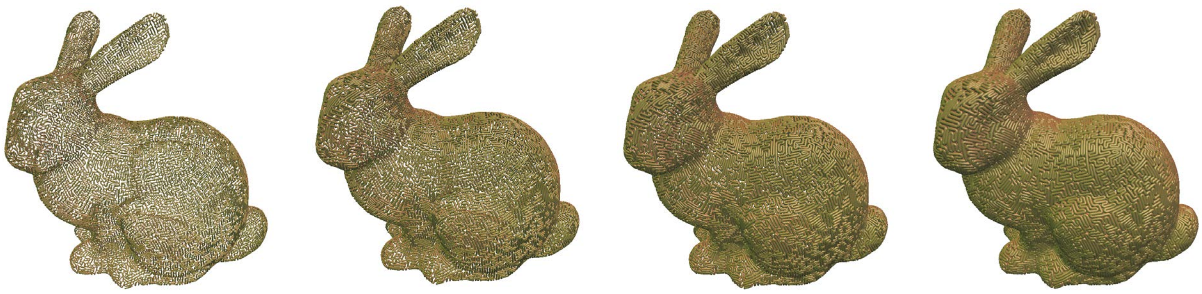
5. Implementation and Results

We have developed a system that converts polygonal meshes to surface filling curves. We provide s and t parameters to control the size of trapezoids. A user can interactively



Sparse Dense Sparse Dense

Figure 10: *Buddha model covered with sparse and dense ribbons. These images are obtained by using variable diameter method. Back-face parts of the ribbon are not drawn for cleaner images. The original quadrilateral mesh is obtained by wave-based anisotropic quadrangulation.*



Thinnest Thin Thick Thickest

Figure 11: *A Bunny quadrilateral mesh covered with constant diameter yarns. Note that if all faces of the original mesh are approximately same size as in this example constant diameter method can densely cover the surface without significant self intersection. Back-face parts of the yarns are shown. The original quadrilateral mesh is obtained by Quadcover method.*

change the thickness of ribbons and yarns by changing the parameters s and t. A very dense covering ribbon is obtained with value $s \approx 1$ and $t \approx 1$. Small values of s and t provide sparse covering. All the images in this paper are direct screen captures from the system; they were created in real-time. Our variable diameter method guarantees that the sizes are relative to the underlying triangles. Therefore, the actual widths of ribbons are different in different parts of the mesh.

If mesh models are created by a good quadrangulation scheme such as Quadcover method [KNP07] Mixed-integer quadrangulation [BZK09] wave-based anisotropic quadrangulation [ZHLB10], then even constant diameter method can cover the surface without significant gaps as shown in Figures 10, 11 and 12. This is mainly because such quadrangulation methods creates almost-regular quadrilaterals. Moreover, there are only limited number of non-4-valent vertices. Constant diameter method can also cover triangular meshes densely if triangles are regular and vertex valences are mostly 6.

6. Conclusions and Future work

In this paper, we presented the concept of surface filling curves, which are constructed from manifold mesh surfaces. We presented an algorithm that can convert any manifold mesh surface to a closed 3D curve that follows the shape of the surface. We have developed two methods to construct corresponding 3D yarns, which can be turned to physical sculpture by using 3D printers. Since 3D prints of such yarns will use much less material than original 3D shapes and 3D printing charges are based on volume of the printed object, 3D printing these yarns can be much more economical than 3D printing original models. For further applications to be used for sculptors, there is a need for simpler user interfaces

Thinnest Thin Thick Thickest

Figure 12: *Another quadrilateral mesh covered with constant diameter yarns. Note that if all faces of the original mesh are approximately same size as in this example constant diameter method can densely cover the surface without significant self intersection. Back-face parts of the yarns are shown. The original quadrilateral mesh is obtained by mixed-integer quadrangulation.*

that allow sculptors to design surface curves. It is also possible to color the different portions of the curve to obtain texture mapping effect.

There are two ways to obtain different curves for a specific shape: (1) To use a different quadrilateral mesh that approximate the shape and (2) for a given quadrilateral mesh to use a different Hamiltonian cycle. As discussed earlier, even for a given mesh there are exponentially many ways to form these curves since with probability 1, there are 2^{F-1} Hamiltonian cycles for any given M where F is the number of faces of mesh M [XACG10].

We are grateful to Wenping Wang, Li Yupei, Muyang Zhang, Jin Huang, Xinguo Liu, Hujun Bao, David Bommes, Henrik Zimmer and Leif Kobbelt for providing good quality quadrilateral models. We also want to thank anonymous reviewers for their helpful suggestions. This work partially supported by the National Science Foundation under Grant No. NSF-CCF-0917288.

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