Duotone Surfaces

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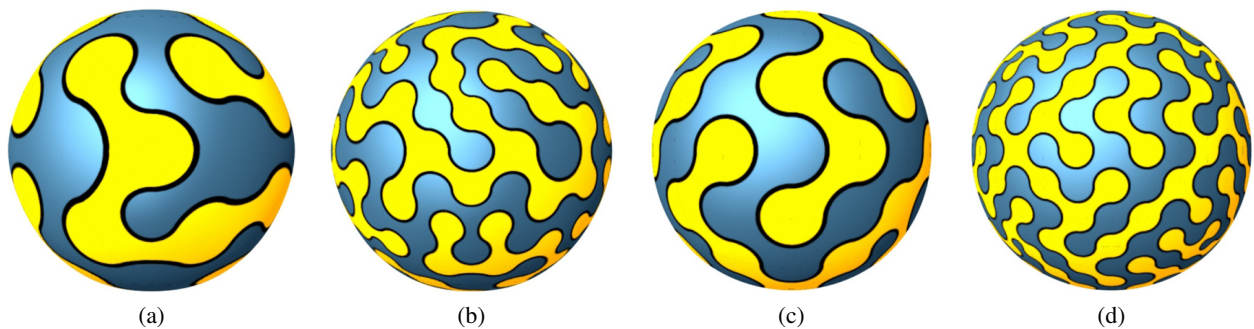
Figure 1: *Duotone surfaces consist of two regions that are bounded by a single curve that covers the surface. These are examples of sphere shaped duotone surfaces. The meshes in (a) and (b) are Catmull-Clark subdivided dodecahedrons, and the meshes in (c) and (d) are subdivided octahedrons.*

Abstract

In this paper, we present a method to divide any given surface into two regions with two properties: (1) they are visually interlocked since the boundary curve covers the whole surface by meandering over it and (2) the areas of these two regions are approximately the same. We obtain the duotone surfaces by coloring these regions with two different colors.

We show that it is always possible to obtain two such regions for any given mesh surface. Our approach is based on a useful property of vertex insertion schemes such as Catmull-Clark subdivision: If such a vertex insertion scheme is applied to a mesh, the vertices of resulting quadrilateral mesh are always two colorable. Using this property, we can always classify vertices of meshes that are obtained by a vertex insertion scheme into two groups. We show that it is always possible to create a single curve that covers the whole surface such that all vertices in the first group are on one side of the curve while the other group of vertices are on the other side of the same curve. This single curve serves as a boundary that defines two regions in the surface. If the initial distribution of the vertices on the surface is uniform, the areas of the two regions are approximately the same.

We have implemented this approach by appropriately mapping textures on each quadrilateral. The resulting textured surfaces look aesthetically pleasing since they closely resemble planar TSP (traveling salesmen problem) art and Truchet-like curves.

Categories and Subject Descriptors (according to ACM CCS): I.3.m [Computer Graphics]: Miscellaneous—visual arts; I.3.7 [Computer Graphics]: Three-Dimensional Graphics and Realism—Color, shading, shadowing, and texture; I.3.7 [Computer Graphics]: Three-Dimensional Graphics and Realism—Fractals
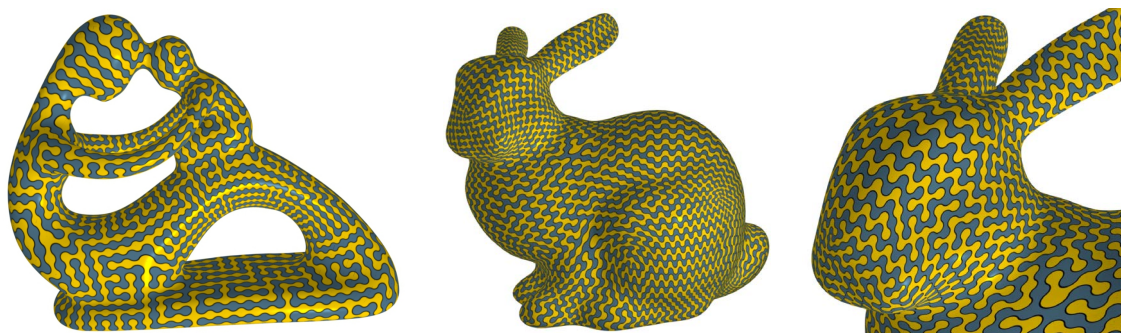
Figure 2: *Fertility and Stanford bunny as duotone surfaces.*

1. Introduction and Motivation

The Jordan Curve Theorem states that any simple closed curve in the plane separates the plane into two regions: the part that lies inside the curve, and the part that lies outside it [PW12]. Although the theorem seems to be very intuitive, the proof is complicated since closed curves can be complicated sometimes such as fractal curves. Many artists observed this property to create artworks over plane by creating interesting curves such as Fractal art, Traveling salesmen problem(TSP) art and Truchet-like curves. Interestingly, Jordan's theorem is only correct for planes. Any single curve on a surface with positive genus does not necessarily separate the surface into two regions.

Recently, Xing et al. [XATC12] developed a method to construct a single curve that covers a given surface. This work is based on Gabriel Taubin's work on constructing Hamiltonian triangle strips on quadrilateral meshes [Tau03]. With this algorithm every connected manifold quadrilateral mesh without boundary can be represented as a single Hamiltonian generalized triangle strip cycle. To construct a single closed curve from this connected triangular strip cycle, one can simply connect centers of triangles in the triangle strip to obtain a single closed curve in 3D. However, this simple closed curve may not necessarily separate the surface into two as indicated by Jordan's Curve Theorem.

In this paper, we present a simple approach to construct simple closed curves that can separate surfaces into two regions (see Figures 1 and 2). We have implemented our approach using Truchet tiles where the boundary curve is not explicitly constructed but appears as the boundary of two regions formed by Truchet tiles. Therefore, our implementation can be considered as an embedding of duotone Truchet tiles over surfaces [Bro08a]. We therefore call our textured surfaces duotone surfaces. However, unlike duotone Truchet tiles our duotone surfaces guarantee only two regions separated by a single curve.

There are two ways to control aesthetic possibilities for duotone surfaces:

- The shape of any given surface can be approximated by a wide variety of meshes. Starting from different meshes,

one can obtain different textures. Examples that show the effect of the structure of the underlying mesh on a spherical shape are shown in Figure 1.
- Even for a given mesh there are exponentially many ways to form these curves. This property provides additional aesthetic possibilities since designers can have additional control over the shapes of the curves.

2. Previous Work

Duotone coloring of plane using Jordan curve theorem is an artistic technique to create planar art using complicated curves. In artistic applications, the most widely used examples of such complicated curves are fractal curves [Man82]. Duotone images using fractal curves are very well-known among mathematicians/artists after Benoit Mandelbrot's seminal work on Fractal Geometry [Man82]. Mandelbrot created many examples of duotone art especially using space filling curves. Mandelbrot also discovered a simple way to treat open space filling curves as closed by assuming they were drawn on a sphere (See Figure 4).

Giuseppe Peano [Pea90] discovered space filling curves. Mathematician and artist Douglas McKenna [McK78], who also created many images in Mandelbrot's Fractal Geometry of Nature, enumerated over 20 million new space-filling recursive designs. Ken Knolton [Kno02] created a portrait of Douglas McKenna using one of the first space filling curves McKenna discovered. The main aesthetical advantage of space filling curves over other fractal curves for creating duotone art is that they result in indistinguishable inside and outside structures as shown in Figure 4.

Robert Bosch and Adrianne Herman invented another curve generation method resulting in interesting duotone plane art, called TSP art [BH04, BH04]. In TSP art we create a set of points representing cities. A traveling salesman who resides in one of the cities wants to visit each of the other cities exactly once and then return home. The salesman would like to visit the cities in an order that will minimize the total length of his tour. One of the most well known and well-studied problems in mathematics, computer science, and op-
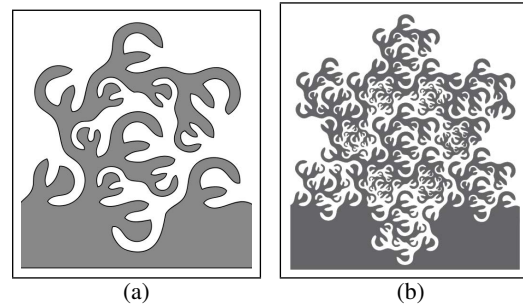
Figure 4: *An example of Mandelbrot's duotone space filling curve art. This particular sequence of images are created by Alexis Monnerot-Dumaine under the pseudo-name Prokofiev in 24 January 2010.*

erations research is determining an optimal itinerary for the salesman in an instance of TSP.

Bosch and Herman noticed that for interestingly placed city locations, the piecewise curve showing the salesman's itinerary looks artistic. They used points on a grid to create an original artwork. This method was simple but required large number of dots to produce a decent picture because the dots tended to clump together. Craig S. Kaplan [KB05] used weighted Voronoi stippling to create positions of the cities. With weighted Voronoi stippling, using substantially fewer dots, it is possible to obtain a more organic appearance. Another advantage of the optimal tours is that they are guaranteed to be closed simple curves. Therefore, TSP art are always colored using two colors to create duotone plane art.

Another related work is Truchet tiles, which was originally introduced by Sebastien Truchet as all possible patterns formed by tilings of right triangles oriented at the four corners of a square [Tru04]. The work related to ours was introduced by Clifford A. Pickover [Pic89]. He created various artworks using a single tile consisting of two circular arcs of radius equal to half the tile edge length centered at opposite corners. The two possible orientations of this tile, and tiling the plane using tiles with random orientations gives visually interesting curves called Truchet curves [Bro08b]. Multiple Truchet curves are also used to create duotone colored plane artworks [Bro08a]. This method is not based on Jordan's curve theorem, but instead it uses a property of Truchet curves to obtain multiple regions that can still be colored using only two colors.

In this paper, we introduce duotone surfaces that can be considered as embedding duotone plane art such as TSP or Truchet art to surfaces. Our approach is based on the construction of a single curve on a surface that can separate the surface into two regions. With this property, resulting surfaces can always be colored by two colors. In terms of visual aesthetics, our results most resembles duotone Truchet pla-

Figure 3: *Buddha as a duotone surface.*

nar art. On the other hand, there are also strong visual similarities with TSP and fractal art. For instance, although they are not strictly self-similar, our curves cover the 2-manifold surfaces in similar manner as the space filling curves cover the planes. Unlike TSP art, our duotone surfaces do not guarantee to provide the shortest route, but they visually resemble random TSP art.

Our approach is based on the fact that it is always possible to construct a triangular mesh with an associated Hamiltonian cycle in linear time from a quadrilateral manifold mesh. The construction algorithm simply splits each quadrilateral into triangles and flip edges until the triangles are ordered into a single strip [Tau03]. The process is especially simple if the vertices of initial quadrilateral mesh are 2-colorable. Such 2-colorable quadrilateral meshes can be obtained by some subdivision schemes such as Catmull-Clark [CC78] and dual of Simplest [PR97] subdivisions. 2-colorable meshes are also key to obtain duotone surfaces. In the next section, we provide our methodology.
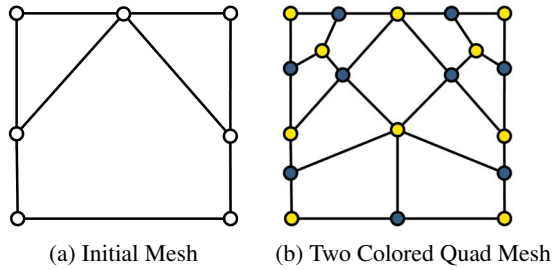

(a) Initial Mesh   (b) Two Colored Quad Mesh

Figure 5: *Conversion of a given mesh into a 2-colorable quadrilateral mesh by using a subdivision schemes such as Catmull-Clark.*

3. Methodology

Our algorithm for generating duotone surfaces consists of four stages.
(1) Convert the input mesh to a 2-colorable quadrilateral mesh.
(2) Color the quadrilateral mesh with two colors and assign textures to its faces.
(3) Connect the disconnected regions on the surface.
(4) Convert the modified two-colorable mesh into a subdivision surface to obtain G^1 continuity.
The formal and more detailed structure of our algorithm is given in Algorithm 1.

3.1. Conversion to a 2-colorable Quadrilateral Mesh

To obtain duotone surfaces, we need 2-colorable quadrilateral meshes. Fortunately any given mesh can be converted into a 2-colorable quadrilateral mesh through remeshing as mentioned earlier. In this paper, we use Catmull-Clark subdivision to obtain 2-colorable quadrilateral meshes. Figure 5

Algorithm 1 Two Region Duotone Surface Construction.

1: Convert input mesh into a 2-colorable quadrilateral mesh, $G = (V, E)$, by using a subdivision schemes such as Catmull-Clark.
2: Color the vertices in V to either BLUE or YELLOW such that no edge exists in E whose end vertices have same color. Say, $U_0 = \{v \in V \text{ and } color = BLUE\}$ and $U_1 = \{v \in V \text{ and } color = YELLOW\}$.
3: Assign a Truchet tile(texture) to each quadrilateral face of G such that the texture is consistent with vertex colors
4: **if** All faces in G are now like 6(b) **then**
5:      Mark the mesh indicating the same.
6: **else if** All faces in G are now like 6(c) **then**
7:      Mark the mesh indicating the same.
8: **end if**
9: **while** Any disconnected vertices in U_0 or U_1 **do**
10:      Pick a face which is not of target triangulation(marked in previous step).
11:      Assign the other Truchet tile to this face such that the texture map is consistent with its new vertex colors.
12: **end while**
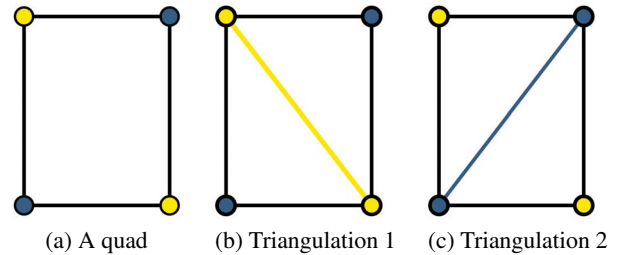13: Convert the polygonal mesh into a subdivision surface to obtain G^1 continuity using Maya for final rendering.

(a) A quad   (b) Triangulation 1   (c) Triangulation 2

Figure 6: *Triangulations of a quad face.*

illustrates the remeshing scheme of Catmull-Clark subdivision, called vertex insertion. As shown in figure, the vertex insertion scheme preserves original vertices of the mesh, which we call vertex-vertex, subdivide each edge by inserting a new vertex in the middle of each edge, which we call edge-vertex, and insert a vertex in the middle of each face, which we call face-vertex. It also inserts edges between every face-vertex and its edge-vertices. In the figure, edge-vertices can be labeled with dark blue color and rest of the vertices can be labeled with yellow color.

3.2. Texture Map Assignment

The underlying graph of a 2-colorable quadrilateral mesh is bipartite [Wei12]. In other words, the vertices are now divided into two disjoint sets U_0 and U_1 such that every edge connects a vertex in U_0 to one in U_1. Moreover, the diagonal
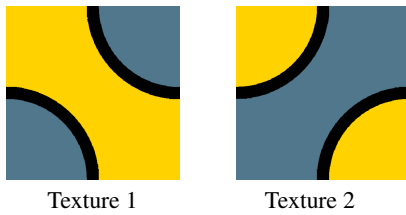
Figure 7: *Texture maps that can cover vertices as defined in triangulations 1 and 2. These particular textures are called Truchet tiles which are used to create duotone planar art.*

vertices of each quadrilateral of the mesh are in the same set, i.e. they have the same label as shown in Figure 6(a).

Our goal is to cover this mesh with a texture in such a way that vertices in U_0 will be colored yellow and vertices in U_1 will be colored blue. For every quadrilateral, there are two possible ways to assign a texture: there can be a connection either between two yellow vertices or two blue vertices. These two possible cases can be conceptualized as two possible triangulations of a quadrilateral as shown in Figures 6(b) and 6(c). The choice of triangulation of a given quadrilateral uniquely defines how to texture map that particular quadrilateral by using textures such as the ones shown in Figure 7.

If we randomly triangulate all quadrilaterals and apply textures based on triangulations, we most likely obtain a two-colored surface that consists of disconnected regions (see some examples in Figure 8). Such random triangulations correspond to the embedding of duotone Truchet planar art to surfaces. Our goal in duotone surfaces is to connect all disconnected regions in the same color. The next section presents how to obtain such duotone surfaces.
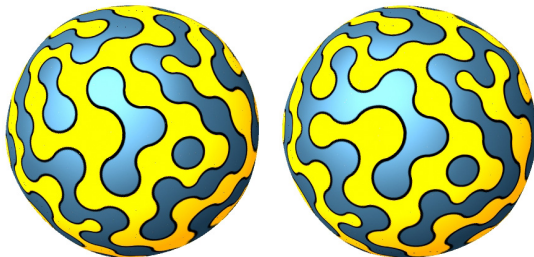

Figure 8: *Duotone surfaces with disconnected regions. Our goal is to make all regions to be connected.*

3.3. Combining Disconnected Regions

Note that it is possible to view the triangulated mesh as a graph that consists of three subgraphs: (1) The original bipartite graph; (2) the yellow graph that connects all vertices in U_0, e.g. the graph that consists of only yellow edges; (3)

the blue graph that connects all vertices in U_1, e.g. the graph that consists of only blue edges. If both blue and yellow graphs are connected, the corresponding texture map will consists of two completely connected regions as we want. On the other hand, if only one of them is connected, there will be disconnected regions in the other one. For instance, Figure 9(a) illustrates an extreme case in which the yellow graph is connected allowing the yellow region to be connected, but the blue graph consists of isolated vertices which resulted in isolated blue regions.

For the surface of a 2-colorable quadrilateral mesh to have only two regions, we require both blue and yellow graphs to be completely connected. This means that neither of these graphs can have a cycle since a cycle in one graph makes the other one disconnected. Thus, both graphs must be spanning trees covering all yellow and blue vertices respectively. If one of these graphs is a spanning tree, the other one is also a spanning tree[1]. Therefore, it is straightforward to obtain duotone surfaces as shown in Figure 9(b).
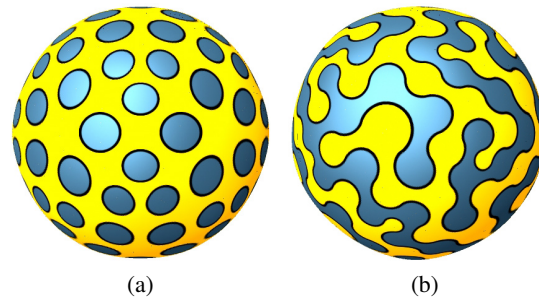

(a)   (b)

Figure 9: *Two duotone surfaces that exhibits completely different behavior. In (a) yellow graph is completely connected, therefore yellow region is connected. On the other hand, blue graph consists of only disconnected vertices, therefore it produces individual circles on surface. In (b) both yellow and blue graphs are trees resulting two connected regions.*

Theorem 1 For a given bipartite graph, say U_0 and U_1 are the two edge disjoint vertex sets and Y, B are Yellow and Blue graphs respectively. If one of the Y/B graph is a spanning tree, then the other is also a spanning tree.

Proof Assume B is not a spanning tree when given that Y is a spanning tree. B is not a spanning tree $\Rightarrow \exists$ at least one cycle in B. A cycle in $B \Rightarrow \exists$ a set of connected edges which is $\subset E(Y)$ but are isolated. Existence of a set of isolated edges in Y contradicts our given hypothesis that Y is a spanning tree. Hence, proved. Similarly, the converse is also true. □

In practice, constructing Hamiltonian triangle strips on quadrilateral meshes is sufficient to construct both yellow and blue trees. Taubin [Tau03] presents a simple linear time and space constructive algorithm, where each quadrilateral face is split along one of its two diagonals and the resulting triangles are linked along the original mesh edges. The triangles are flipped until we obtain a Hamiltonian strip. The
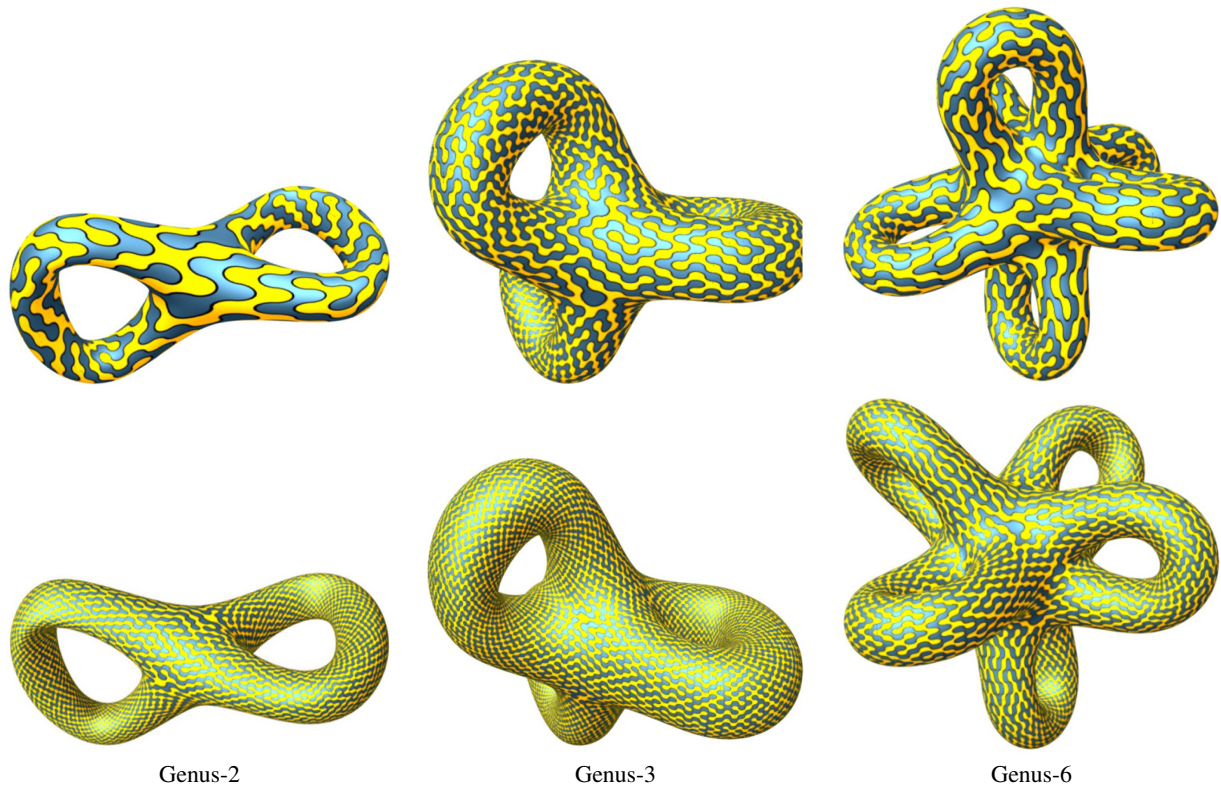
Figure 10: *Positive-genus duotone surfaces.*

Hamiltonian strip is actually the representation of the curve that serves as the boundary of blue and yellow regions. The diagonal edges in the resulting triangulation consists of two spanning trees. This hamiltonian strip is not unique. In fact, there are 2^{F-1} Hamiltonian cycles for any given M where F is the number of faces of mesh M with probability 1 (see [XACG10] for a related problem). Using this property, we can control the resulting surface coloring by altering the number of branches. We prefer high branch count for both yellow and blue trees which result in more wavy boundary between two regions [Tau03, XATC12].

3.4. Conversion to Subdivision Surface

One final issue is that direct texture mapping of polygonal meshes results in G^1 discontinuities since a polygonal mesh is not G^1 continuous across the edges. We simply turn the polygonal mesh into a subdivision surface. Note that Catmull-Clark subdivision surfaces are already G^2 continuous everywhere except extraordinary vertices. As our texture maps have same color around vertices, discontinuous regions around extraordinary vertices cannot be visible. On the other hand, the original Truchet textures are only G^1 continuous in edge boundaries i.e. the two circles boundaries meet in the same point with the same tangent, but the centers of

the circles are not the same (see Figure 7). Thus, we obtain only G^1 continuous texture map although the surface itself is G^2 continuous in edge boundaries. As shown in the figures 11, it can be seen that G^1 continuity is sufficient to obtain good looking results.

4. Implementation and Results

To obtain duotone surfaces, we have only implemented texture mapping as a stand alone software using C++. The initial Catmull-Clark subdivision is done using publicly available software. The resulting mesh is exported as a non-textured .obj file. Our texture mapping software reads this .obj file and assigns appropriate texture and texture coordinates to each quadrilateral of the 2-colorable quadrilateral mesh. Now the textured mesh is exported as .obj file. We then import this textured mesh into Maya [Aut10] and turn it to a subdivision surface since Maya provides good quality subdivision surface in realtime [Sta98]. All images in this paper are rendered in Maya as subdivision surface using default lighting. Figure 10 shows several examples of duotone surfaces that are obtained by this process and rendered by Maya. To obtain higher frequency images, we simply obtain denser polygonal meshes using subdivision as shown in Figure 10. We assume that the meshes do not have high aspect-
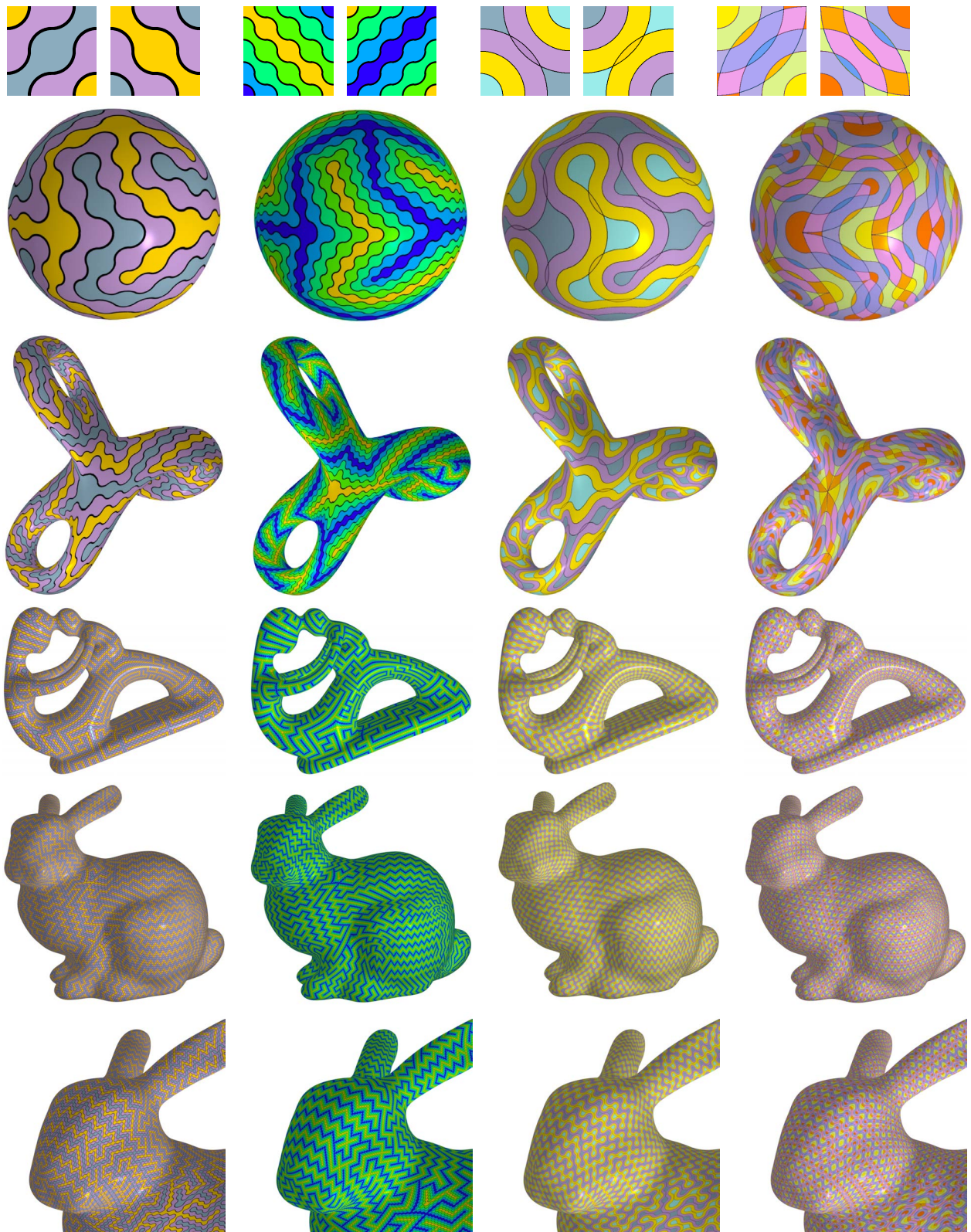
Figure 11: *The top row shows four possible tiles that can be used to obtain more colorful versions of duotone surfaces. Duotone surfaces in each column are created using these tiles.*

ratio or concave quadrilaterals. Such quadrilaterals might result in visually uninteresting results. Since we could not find references to any methods doing similar work, we could not compare the results against existing standards.

4.1. More Colorful Examples

Strict Truchet tiles are not the only one that can be used for texturing duotone surfaces. It is in fact possible to create a wide variety of aesthetic results using more colorfully designed tiles such as the ones shown in Figure 11.

5. Conclusions and Future work

In this paper, we presented the concept of duotone surfaces, which can be obtained from any manifold mesh surface by first subdividing and then texture mapping appropriate tiles. Our duotone surfaces consists of two regions that are visually interlocked. Their boundary curve covers every part of the surface by meandering over the surface. Moreover, the areas of these two regions are approximately the same. We have implemented this approach by texture mapping the two texture maps appropriately on each quadrilateral. The duotone surfaces can also provide sculpting opportunities. For instance, the two regions on the duotone surface can be obtained by cutting the surface into two 2-manifolds with boundaries. To create a sculpture, these two manifold with boundaries can be turned to solid shapes which can be interlocked together to form the original shape.

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