Computational Aesthetics in Graphics, Visualization, and Imaging (2009)
O. Deussen and P. Hall (Editors)

Aesthetic Placement of Points Using Generalized Lloyd Relaxation

Oliver Deussen

Department of Computer Science, University of Konstanz, Germany

Abstract

In this paper we describe a computational method for producing aesthetically pleasing distributions of disks on a canvas. The positions of the disks are initially given at random and are moved into interesting configurations by means of a local optimization routine. The configurations are computed by a Voronoi-cell based optimization algorithm (Lloyd's relaxation method). We extend this method in a way that not only evenly spaced but also clustered point sets can be produced. This is done by inverting the iterative step of the optimization algorithm. We define an energy term and show that for a certain amount of energy interesting configurations appear. This is evaluated in a small user study.

1. Introduction

Distributing simple objects such as disks on a canvas is an introductional task in art and design, it is part of many text books and has inspired many artists for creative solutions. Though there exist many rules of thumb for such placements, computational methods are rarely known. In this work we try to find quantitative measures for aesthetically distributed disk sets - a very simple special case of the general object placement problem: given a set of points that are represented by disks with an uniform radius and color, we create aesthetically pleasing distributions on a rectangular canvas by a simple optimization algorithm.

We use a Voronoi-cell based local optimization scheme for moving the points that are initialized at random positions. This optimization, the so-called Lloyd's method, moves the points towards the center of gravity of their associated Voronoi cells and creates more and more even point distributions. In the stationary case this often results in a quasi hexagonal distribution which is quite uninteresting in the eyes of the average viewer.

We will see later that aesthetically interesting point distributions have a certain amount of irregularity, of clustering. To introduce irregularity we extend the Lloyd's Method by allowing the points also to move into the opposite direction - away from the center of gravity - thus inverting the optimization. Doing so, the points form small sets of clusters.

Figure 1 shows typical disk sets. An initial distribution is shown in (a), the stationary configuration after a number of optimization steps is given in (b), subfigure (c) shows a clumped configuration while in (d) a set is shown that is a compromise between regularity and clustering.

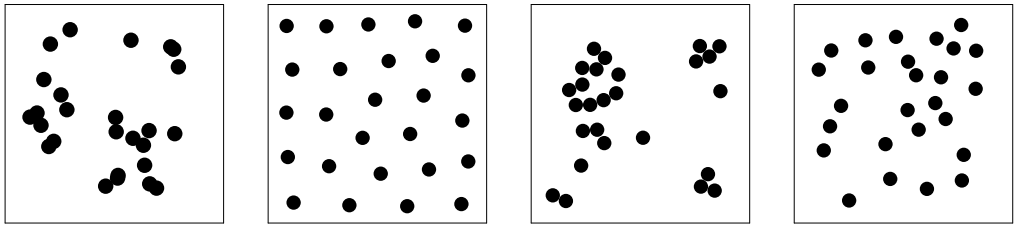
In a small user study we show that for a certain amount of irregularity and clustering aesthetically pleasing sets are created. It has to be noted that this is only one very special method for creating such clusters, many others can be found. However, our algorithm has the advantage allowing the user modifying the amount of clustering by changing just a single energy parameter. Aesthetics in this special case is maintaining a certain energy level for the point distributions.

After reviewing related work from graphics and arts we will describe Lloyd's relaxation method and our extension. The formulation of the tension energy is given and the resulting sets are evaluated. A number of examples are given.

2. Related Work

Distributing elements is a basic task in design. Visual weight and the related rules play a major role in text books on art, design, architecture (see, i.e., [Lau99]). Unfortunately, the mechanisms of placing elements are usually described qualitatively and not in the form of quantitative values or measures. This is due to the relation of design to Gestalt Theory, which is rather nonlinear and also not defined by quantitative rules [EK00, Saw00].

DOI: 10.2312/COMPAESTH/COMPAESTH09/123-128



(a) (b) (c) (d)

Figure 1: *Some distributions: a) initially given points; b) stationary state after optimization; c) clumped distribution achieved with inverting the optimization; d) an "interesting" distribution as the result of an intermediate level of clustering.*

An inspiring work on analyzing art work mathematically was presented by Dodgson [Dod08]. The author analyzes abstract design works of Bridget Riley by measuring the needed amount of randomness for an aesthetic form. This is one of the rare approaches to computationally describe art with random functions.

In the following we will work with so-called Centroidal Voronoi Tessellations. The necessary details are given later; however, a good reference was presented by Du et al. [DFG99] which give a detailed overview. Some important findings regarding interesting configurations of such tessellations and corresponding energy levels are given by [Ger79, New82].

Centroidal Voronoi Tessellations are already used for creating artistic point distributions like stipple patterns [DHvOS00, Sec02]. Here, a random point set is wanted but the nearest neighbor distances should be as uniform as possible. Another application for such sets is sampling [KCOD-Loc] and numerical integration [SB80, Smi74]. In both cases their unique spectral properties (blue noise characteristics) are the reason for their usage.

3. Voronoi Tessellations and Lloyd Relaxation

Let $S = s_1, .., s_n$ be n points in $\mathbb{R}^2$. The Voronoi region $V(s_i)$ (also called Voronoi cell) of a point is the area in $\mathbb{R}^2$ in which each point is closer to $V(s_i)$ than to any other point in S. The regions form a tessellation of $\mathbb{R}^2$ in that they are pairwise distinct and jointly covering the entire $\mathbb{R}^2$. The tessellation is called the Voronoi diagram $VD(S)$ of S. Since a tessellation of the entire $\mathbb{R}^2$ has open Voronoi regions for some points, we close such regions by intersecting these regions with a rectangular frame that encloses the given points. This frame is our canvas on which the points are distributed.

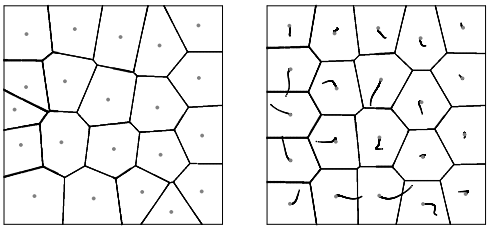
An *ordinary Voronoi Tessellation* is a tessellation that uses the Euclidian metric as a distance function. Many other distance functions are possible. However, for the purpose of this work the Euclidian distance is sufficient since it is a "natural" distance function with an intuitive relation between value and perceived distance.

A *Centroidal Voronoi Tesselation (CVT)* is a (in our case ordinary) Voronoi Tessellation with the additional property that every point s_i lies in the centroid of its Voronoi region $V(s_i)$. Such a tessellation is shown in Figure 2(a). The points are still almost at random but now with much less variance in their point to point distances.

Such a tessellation and corresponding distribution of points can be achieved by applying the *Lloyd relaxation* to the points. In each step of this iterative algorithm, each point is moved into the centroid of its Voronoi region. If c_i is the centroid of a Voronoi Region the movement is given by

$$s_i^{(t+1)} = s_i^{(t)} + \alpha\left(c_i^{(t)} - s_i^{(t)}\right), \tag{1}$$

where $\alpha \in (0,1]$ determines the convergence. The iteration is repeated until the movement of the points is below a given threshold. Figure 2(b) visualizes the movement during such a relaxation.



(a) (b)

Figure 2: *a) Voronoi Diagram; b) Movement of points during relaxation.*

The CVT minimizes the following energy function that measures the compactness of the Voronoi Regions (see [DFG99]):

$$F_v(S, V(S)) = \sum_{i=1}^{n} \int_{V(S_i)} ||x - s_i||^2 dx. \tag{2}$$

This energy function sums up the integral of all quadratic

distances of the points in a Voronoi Cell towards the corresponding point s_i. This is minimal for compact regions with almost uniform aspect ratio, thereby approximating hexagons [DFG99]. Furthermore, it implies an almost uniform distribution of point-to-point distances since these distances are maximized for compact regions.

F is depending from the number of points [DFG99]. Since in our experiments we want to compare sets with various sizes we use a normalized value which is constant for all sizes of sets:

$$F = |S| \cdot F_v. \tag{3}$$

The Lloyd relaxation minimizes Eq. (2) and therefore can be used as a local optimization method for F. It moves the points into a distribution with almost uniform point-to-point distance which often converges to a hexagonal distribution. This can be seen in Figure 1(b).

Interestingly, the inverse operation results in a clustered point distribution. In Eq. (2), the parameter α is now set to a value $\alpha \in (0,-1]$, pushing the points s_i away from the centroids of their Voronoi regions.

4. An Extension of Lloyds Relaxation

The central statement of this work is that point distributions are aesthetic or at least "interesting" to the users if they have a certain amount of clustering, of irregularity. The classical Lloyd relaxation always ends up in an almost uniform point distribution –as stated above– which is aesthetically not very interesting. We therefore extend the relaxation in a way that a given amount of irregularity is maintained and can be adjusted by the user. This enables us to evaluate the needed amount for aesthetic distributions.

In our tool the relaxation is performed continuously throughout the whole evaluation by the user, however with a varying $\alpha \in [-1,1]$ that in our case can also be negative. The value is chosen in dependency to F from Eq. (3). The user selects a desired energy value E and the value α is cmputed by

$$\alpha = w_a \frac{F-E}{F}$$

with w_a being a weighting factor ($w_a \in [0.25..1]$). The above formula results in negative values for α if the desired energy is larger than the system and therefore the points are moved apart from the centroids of their Voronoi regions. In any case we ensure $\alpha \in [-1,1]$ to avoid over-relaxation.

In the case of $E < F$ the conventional Lloyd relaxation is performed, the points are moved towards the centroids of the Voronoi Regions and the whole distribution is transformed into a more evenly spaced one. The small values for α reduce the movement of the points if F comes close to the intended energy value E. The same holds for $E > F$. In this case the points are moved away from the centroids with a speed that is also reduced if F approximates the intended energy.

To obtain a better interpretation for the values of F and E, we present all energy values in the following as a percentaged value in the interval between the minimal and maximal energy that is possible for the system on the unit square $(x,y) \in [0,1]$.

According to [Ger79, New82] the *minimal energy* F_{min} is achieved for a hexagonal point distribution and is given by

$$F_{min} = \sum_{i=1}^{n} \int_{V(S_i)} ||x - s_i||^2 dx = \frac{5}{18\sqrt{3}} \approx 0.1604.$$

Please note that the above authors use a unit square $(x,y) \in [-1,1]$ and therefore find a value $F_{v(min)} = \frac{40}{9\sqrt{3}}$ which is 16 times our value.

The *maximal energy* F_{max} is achieved if all points are placed one above the other in one corner of the unit square. In this case all Voronoi regions are empty except one region which now covers the whole unit square. This value is not depending to the number of points and can be obtained by:

$$F_{max} = \int_0^1 \int_0^1 ||x - s_i||^2 dxdy = \frac{2}{3}$$

The energy level E,F are now expressed as a percentage within the interval $[F_{min}, F_{max}]$.

5. Implementation and results

For the visual display, the points S are represented by filled disks with a given size. This size is adapted by the system if the number of points is changed by the user. We ensure that the area of the discs is always a fixed fraction of the overall area.

The size of the visual representation must be integrated into the relaxation since the Voronoi Regions might shrink below the size of the disks which causes touching points that are visually distracting. In the implementation we avoided this by adding a force that pushes points away from each other if their disks touch.

Figure 3 shows typical point sets for a low energy level of $E = 15\%$ values. All systems are stable and are created within a few relaxation steps. Figure 4 shows sets of the same size for a high energy level of $E = 45\%$. In this case the point sets are more clustered, if higher energy levels are used the points tend to cluster too much.

A value $E \in [30..45]\%$ seems to produce nice point distributions for sets with a wide range of sizes. Some of the produced systems can be seen in Figure 5. These systems are stable and can be computed interactively with the accompanying applet. However, it has to be noted that these results are only valid if the size of the disks is related to the overall area (see below). If the size is changed manually the sets visually tend to be more regular or clustered.

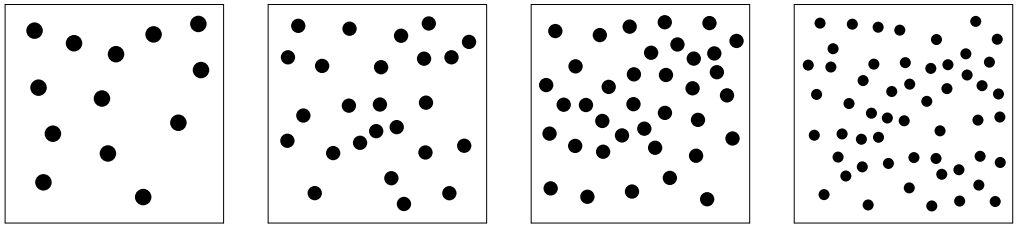


Figure 3: *Point distributions with $E \leq 15\%$ tend to be too regular (13, 25, 53 points)*

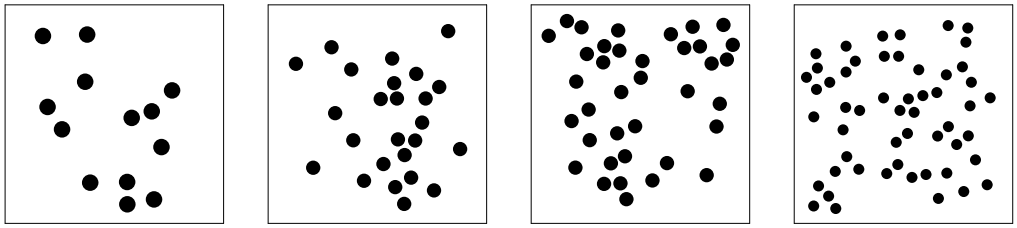


Figure 4: *Point distributions with $E > 45\%$ tend to be too clustered (13, 25, 53 points)*

6. Is This Aesthetic?

We tried to evaluate the aesthetic quality of the point sets by a small user study. First a number of printouts were presented to the test persons and they had to order them according to their aesthetic quality. For each of the energy levels $E = 15, 30, 45\%$ the subjects were given three different disk sets. One group received disk sets with 21 points, another 31 and one group 53 disks each. A total of nine individuals was used for this test, all of them were computer animation experts with an aesthetic knowledge and feeling.

The users in 75% of the cases liked the energy level of 45% most and selected these level at the top three positions. In 20% of the cases $E = 30\%$ was selected. This result is stable for all sets sizes. In 80% of the cases the energy level of $E = 15\%$ was among the last positions, especially for the disk sets of 21 and 31 points. For 53 points the results are not as clear, here we had one low energy set among the top positions and two high energy sets (45%) among the last three positions (for four test persons).

As mentioned above, an energy level of about $E \in [30..45]\%$ seems to be preferred by users since higher levels, especially when disks touch each other, are too clustered. However, the variance of the little test we performed was quite high and sometimes users preferred highly clustered sets, especially for larger point sets.

To analyze these visual effects, Figure 6 shows a point set with increasing energy. An esthetic configuration seems have a relation between the regularity that has to be maintained and the clustering that is needed in order to have some visual structure. We tried to justify these findings by geometric measures and computed the ratio of nearest and second nearest distances for all disks. By averaging this value over all disks we tried to find a measure for the relation between clustering and regularity. However, this measure varies too much for different point sets that are perceived as visually similar.

Another effect that has to be taken into account is the visual dependence of the point set from the sizes of the representing disks. Figure 7 shows how the perception of the set changes when the disks are enlarged manually. The configurations tend to cluster in this case. This is due to the effect that distances between the points are perceived as distances between the borders of the representing disks.

We tried to incorporate this effect by computing the Voronoi Diagram of the disks instead of taking only the points. In this case the Voronoi Regions are shaped slightly different and not convex any more. The Lloyd relaxation converges still converges in this case, though in some rare cases oscillations can occur.

While the maximum energy for such configurations changes only slightly, the minimum energy F_{min} decreases with increasing circle radius. This is due to Eq. (2) in which the squared distance is decreased due to the increasing radius. Figure 8 shows a sample of a Voronoi Diagram for disks and the values F_{min} for different circle radii.

Unfortunately, in this more general case there is no single energy level that creates aesthetically pleasing sets, at least not if the disk size is changed. If the size is increased,

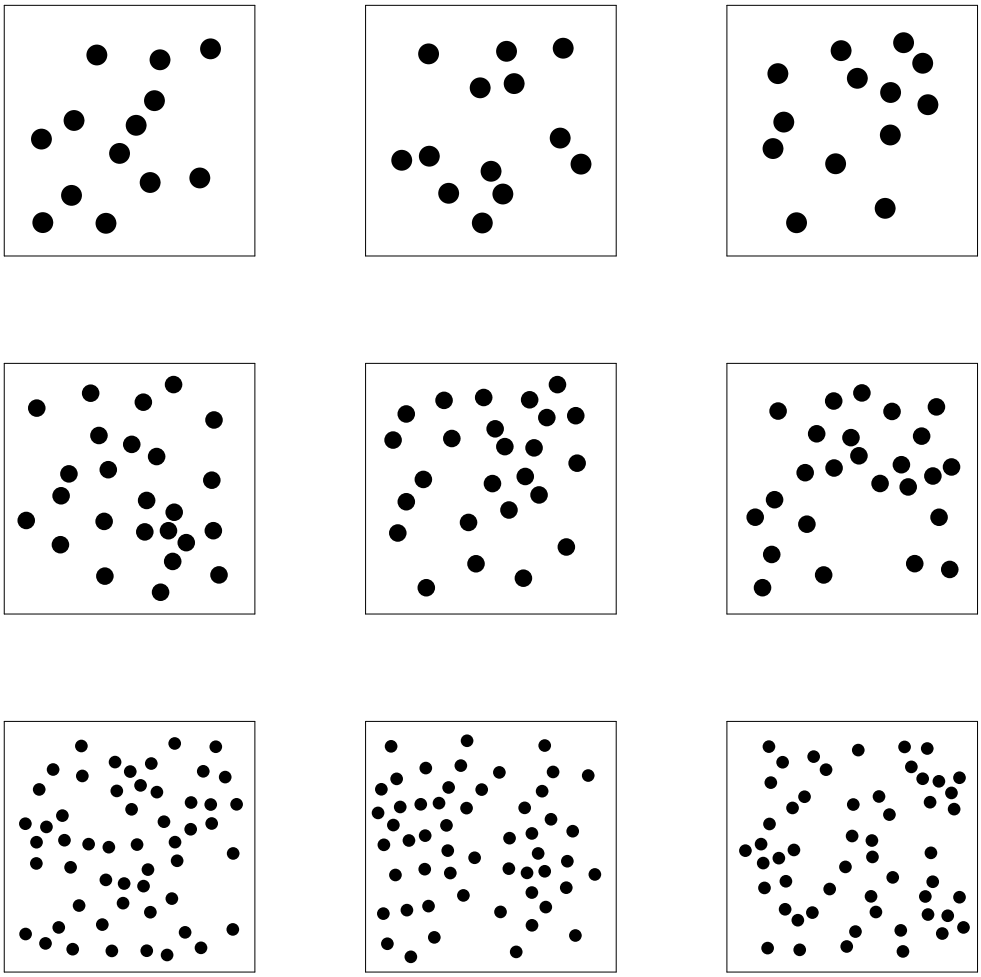


Figure 5: *Point distributions for $E = 30\%$ show a compromise between regularity and clustering.*

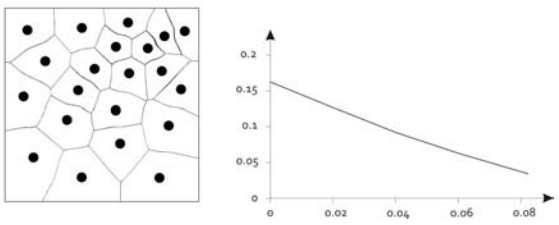


Figure 8: *Voronoi Diagram of a set of disks and values of F_{min} for different disk radii.*

the overall energy level is again reduced due to Eq. (2). To maintain a given level E, the systems pushes the disks away from the centroids and therefore causes a stronger clustering. This behavior is contradictory to what one would expect since when disk radii are increased the set is already perceived as to be more clustered. Instead of balancing out this effect, the shrinking energy level increases it. So far, we do not have a simple solution to this problem.

7. Conclusion

In this paper we described a computational method for creating aesthetic point distributions on a square canvas. We presented an extension of Lloyds relaxation mechanism that enables us to create a variety of point distributions with different characteristics.

If the area of the disks that represent the points are a

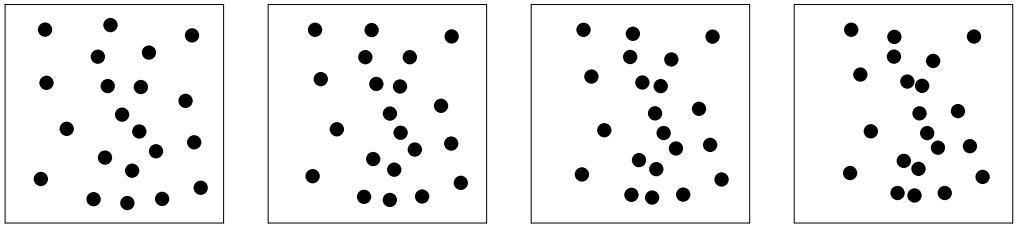


Figure 6: *Point set with increasing energy.*

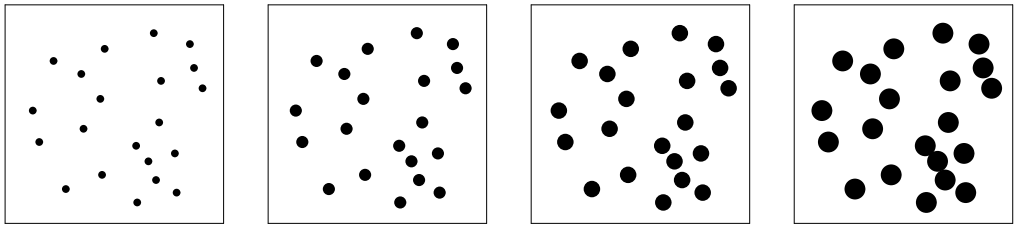


Figure 7: *Point set with increasing size. The configuration tends to cluster when radii grow.*

constant fraction of the overall area of the canvas and are adapted accordingly, we were able to define an energy function that allows us to create aesthetic distributions with a certain amount of clustering and regularity. Though we know that this is only one method for creating such sets, the relatively constant values of the energy function E for aesthetic sets of different size might be a hint for a general rule.

So far we are not able to extend this simple measure to point sets with varying disk radius. This has to be left for future work. Also we want to study other methods of disturbing regular point sets in order to find a balance between regularity and clustering. And last but not least so far no perceptual validation was given why the selected distributions are perceived to be aesthetic.

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