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On Nonlinear Perspectives in Science, Art and Nature

Georg Glaeser

Department for Geometry, University of Applied Arts Vienna, Austria

Abstract

Classic perspectives, i.e., central projections onto a plane, are extremely common in our days. Photos, movies, computer generated animations almost exclusively use this technique. They are linear since straight lines in space appear as straight lines in the image. Nevertheless, humans and animals of all kind have a more complicated method to develop images in their brains. They measure angles, not lengths. Together with nonlinear projections onto curved surfaces, impressions are transformed into spatial imagination.
When it comes to 2D-reproduction of such processes, we need nonlinear perspectives in 2-space. They usually look like fisheye-images, i.e., projections of space onto a plane via a not symmetric, extremely refracting spherical lens. Similar distortions occur when we look out of still water or into reflecting spheres. In fine Arts, the angle measuring was intuitively applied by artists. In geometry, the inversion at a circle (sphere), several models of non-Euclidean geometries and the stereographic projection onto the plane or mappings of the sphere respectively lead to comparable results. We call the latter transformations "secondary" nonlinear perspectives.
Finally, realtime algorithms are presented that transform primary nonlinear perspectives like special refractions into classic perspectives. Therefore, we work with Taylor series (or, if possible, with accurate formulas) and – for speed reasons – with precalculated tables.

Categories and Subject Descriptors (according to ACM CCS): I.3.6 [Computer Graphics]: Nonlinear projections, 3D view deformation, Methodologies and Techniques;

1. Introduction

In this survey paper, we will explain what we mean by "nonlinear perspectives", where they occur and how they can be applied in order to fulfill certain tasks. We will distinguish between primary and secondary perspectives ([Gla05]).

Primary perspective images derive from the intersection of light rays through a projection center with an arbitrary projection surface. The perspective is only linear when we project onto a plane and the rays are not reflected or refracted. Secondary perspectives derive from primary perspectives, e.g., when we project such perspectives from another viewpoint or when we map them onto other surfaces including planes. Even secondary perspectives can be linear in special cases (photos of photos).

Other approaches to nonlinear images of 3-space, e.g., the "multiperspective linear cameras" ([YM04], [RB98]), are not taken into account in this context. A classification of some of those perspectives can be found in [SGN03].

2. Central projections onto planes

In geometry, a central projection of space points P onto an image plane π is also called a *perspective*. The artists of the Renaissance (Brunelleschi, Dürer, Piero della Francesca) developed this kind of depicting reality. In this paper, we will call such images "classic perspectives".

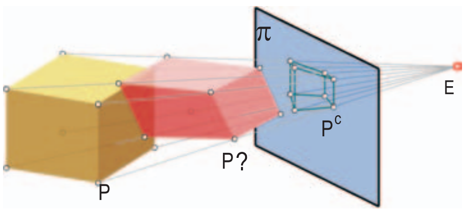

Figure 1: *Classic central projection onto a plane.*

Even if it is not immediately to be seen: Photographic images can be explained as almost exact central projections

(Figure 2 right): All light rays emitted by space points P that hit the front lens are after a detour bundled in an image point P^c (in the ideal case on the light sensitive chip) such that the straight line PP^c runs through a fixed point on the optical axis, called the "focus of the lens system". Therefore, photography is in the ideal case a linear (and primary) perspective – after several refractions that annul each other. Thus, the lens system works like a pinhole camera. Note that natural vision cannot produce linear perspectives, since the light rays hit the curved retina (Figure 2 left).

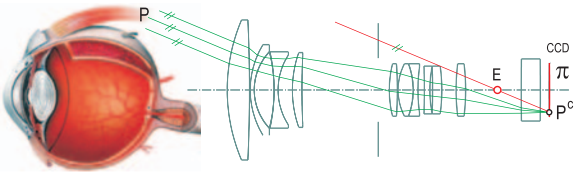

Figure 2: *The eye and the projection by means of a camera.*

Classic perspectives have the advantage that they are *linear*: Straight lines appear as straight lines. The order of a space curve equals the order of its image. Circles, e.g., appear as conics (Figure 3 left). Therefore, the contour of a sphere (a small circle of the sphere) also appears as a conic silhouette.

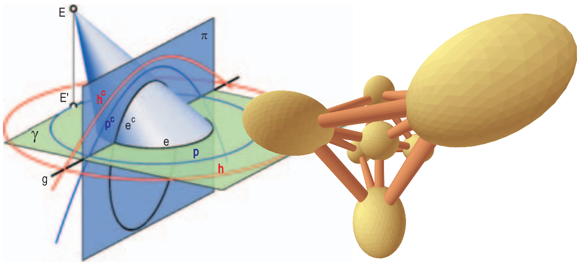

Figure 3: *Central projection of circles and spheres.*

Especially the latter may turn into a drawback. Since the projection onto our retina is non-linear, extreme distortions of sphere silhouettes appear unnatural. We would rather like to see a circle as the silhouette: When we look at a sphere, we immediately try to focus its center, and then the silhouette *is* a circle. There is only one case when we completely agree with an extreme perspective like in Figure 3 right. That is when our "secondary position" with respect to the object is relatively speaking the same as the "primary position" of the photography. (If the image is small like in this publication, we have to be *very* close and position one eye above the principal point. If the image is very large – like in a cinema – this task is easier to fulfill. You should probably sit in the middle of the first row.)

Artists sometimes work with extremely skewed secondary perspectives. Figure 4 (Hans Holbein the Younger, *The Ambassadors*) reveals some macabre detail when it is viewed from an extreme position to the lower left.

Figure 4: *Some extreme secondary views.*

3. Central projections onto concentric spheres

All projections into a lower dimension are not reversible, i.e., without any further information one cannot reconstruct space from one image only (Figure 1).

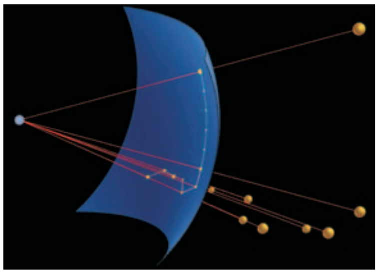

Figure 5: *We measure the angles, not distances*

We now consider a central projection onto a sphere around the projection center. Figure 5 shows the projection of some objects – in this case the stars of *ursus major* that lead to the *big dipper*. The projection sphere is the "sky sphere". No viewer on earth can judge the actual distances of the single stars – even brightness is no criterion. The only thing we can measure, are the angles of the light rays.

Figure 6: *The sphere is not developable ...*

Figure 7 illustrates that, when we stay steady and can only roll our eye balls, measuring angles is the only estimation we can do. Then two parallel lines (e.g., AA_0 and BB_0) will have different "distances", and their images will be curved: On the projection sphere, we have two parts of great circles.

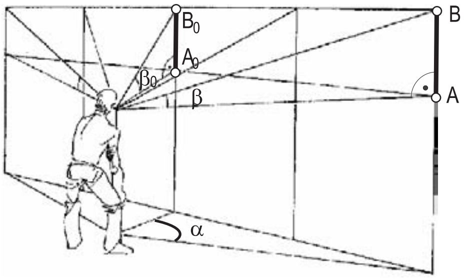

Figure 7: *Different angles for equal distances*

Since the sphere is not developable (Figure 6), it is now more ore less a matter of taste how we map the contents of the spherical image into 2-space. In any case, we then get a secondary nonlinear perspective. (For dozens of samples see `http://www.geometrie.tuwien.ac.at/karto/`.)

Figure 8: *Degas paints the interior of a small room.*

Edgar Degas did it in his painting *Intgerior at Menil-Hubert* (Figure 8) like Guido Hauck suggested it from a scientific point of view some years later ([Hau10], [PPC97]): The Cartesian coordinates in the drawing plane are the spherical coordinates, i.e., angles (Figures 9 and 29 right).

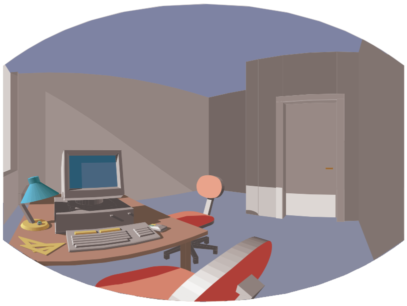

Figure 9: *A small room via computer.*

If we project classically onto a plane in between the space points and the projection center, we get a perspective that is related to the curved projection onto the sphere around the projection center. In fact, if the viewer is located in the projection center, no difference can be seen between the two perspectives, since all angles that can be measured are equal.

Figure 10: *Curved perspectives can be stitched together.*

Hauck's transformation can be used successfully in order to stitch several photographs together (Figure 10 right). This only works perfectly, however, when we know the absolute value of one angle (e.g., the filed-of-vision-angle). When the main projection ray is horizontal, vertical edges of the scene will appear as vertical lines (Figure 10 left), whereas general straight lines will appear as curves.

4. Central projections onto arbitrary surfaces

Figure 11: *Central projection onto an arbitrary surface*

Instead of projecting onto a plane or a sphere around the center, we can project space on arbitrary surfaces. We can also do that as follows: First we create a classic perspective, i. e., a photo, and then we project that photo from the same center onto the curved surface. The fact that a (single-eyed) viewer can only measure angles, allows to create illusions. When a viewer comes close to the original projection center, his or her view of the curved perspective will converge to the view of the photo (Figure 11).

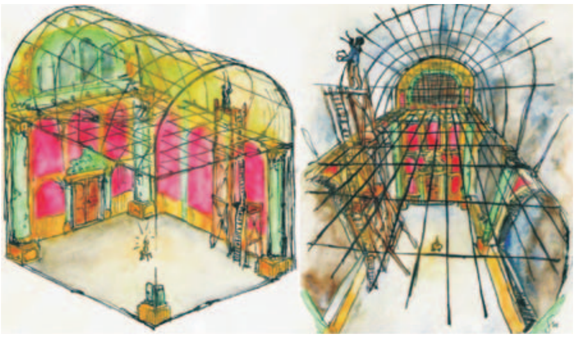

Figure 12: *Pozzo projected a planar grid.*

In that way, the baroque artist Andrea Pozzo painted perfect curved perspectives on the curved ceilings of huge halls. For a viewer who stands in the center of the hall, these paintings appear like classic perspectives of a virtual reality (sky, angels, huge columns, etc.). In order to get the correct painting, Pozzo worked with an auxiliary planar grid in the "photo" (Figure 12 left) of the virtual scene. He materialized this grid with cords which he projected onto the ceiling by means of candle light from the viewer's position (Figure 12 right).

Figure 13: *The kind of projection surface plays little role.*

For the viewer down in the hall, it was then hard to distinguish, what kind of surface the ceiling is. The "photo" suggests a virtual space (Figure 13).

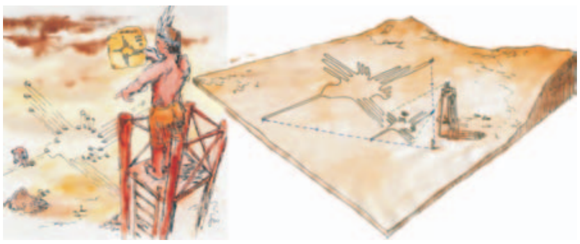

Figure 14: *How the Nazcas may have painted their images.*

The huge sand carvings ("paintings") of the aboriginal people in Peru (the Nazcas) may have been done as indicated in Figure 14: A person standing on a tower (left) gives commands to some helpers in the landscape, directing them until all the angles to the stones they drop equal the angles to the points of the image in his hand. In a second step (right), the dropped stones are moved along straight lines from an arbitrary center *C*. The new distance from the center is a constant multiple of the original distance. After these two steps, a person on a virtual tower of multiplied height (e.g., in a plane or a balloon) will see the image as the person on the tower saw both his sketch and the dropped stones. In reality, however, we deal with a curved perspective in a landscape. Thus, from the "wrong" point of view, the image will appear skewed.

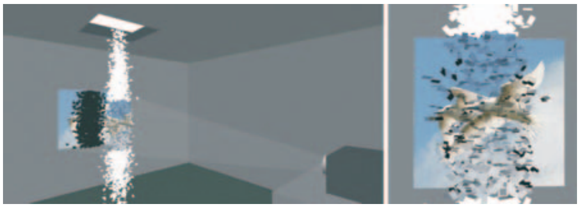

Figure 15: *Projection on a discontinuous surface.*

An experiment to again explain this fact may be the following: A beamer projects a classic (in the ideal case again primary) perspective onto the wall. Viewers close to the beamer will acknowledge this secondary perspective as realistic. When now tiny paper sheets fall down from the ceiling (Figure 15), the image will be projected on a discontinuous surface. The viewer very close to the beamer will, however, still claim that the perspective is OK.

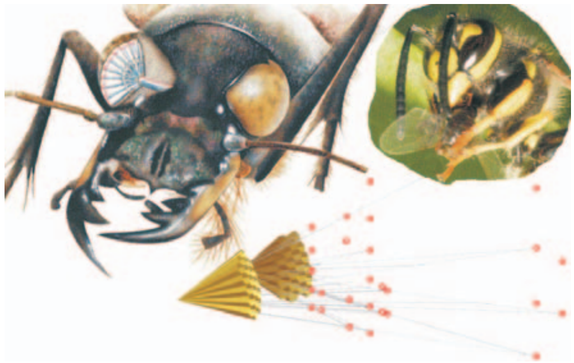

Figure 16: *Measuring angles may lead to perfect 3D-vision.*

Figure 16 illustrates how insects measure angles via hundreds of *ommatidia* (narrow cones as parts of their complex eyes). The outer surface of the eyes need not be spheres – the result is always the same. Together with the second complex eye, the animal has partly perfect 3D-vision in the close neighborhood in front: Many of the cones' axes intersect at fixed points in space that are stored "via hardware".

Human eye balls roll spherically in order to focus on a space point. The different "roll angles" allow to reconstruct the point in space. This requires *firmware* – and thus brain work. However, the principle is the same.

5. Refractions as nonlinear primary perspectives

Photos usually come as close to classic perspectives as possible. This is – technically speaking – a complicated procedure that requires a whole lens-system (Figure 2 right). Sometimes we need (or want to) use different lenses. This immediately leads to curved perspectives.

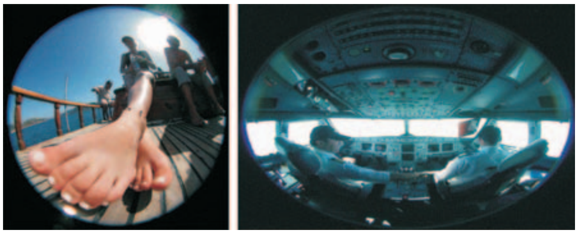

Figure 17: *Fisheye-photographs.*

Figure 17 shows "fisheye perspectives", created by a strongly spherically curved objective (the image to the right is stretched). Such a perspective stems from a classic perspective and is then skewed radially from the principal point. Thus, straight lines that intersect the principal ray will still appear as straight lines. Circles with the principal ray as axis will also appear as circles.

The word fisheye perspective is perfectly suitable for the latter perspectives: When a fish looks out of still water, similar distortions occur (Figure 18), although the light rays are refracted at a plane. Due to total reflection, the whole image of the outside space fits into a limiting circle (Figure 19 left and middle, [Gla05]).

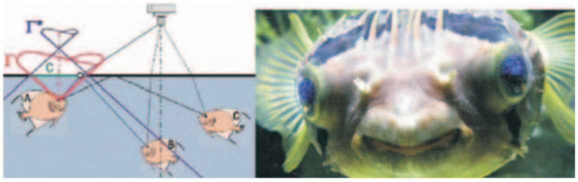

Figure 18: *Refraction and total reflection on the surface.*

By the way, the spherical curvature of the fishes' cornea is a method to diminish additional refraction on the cornea. Fishes that hunt at the surface (Figure 19 right, four-eyed fish) need therefore two differently curved corneas.

Generally speaking many living beings work primarily

Figure 19: *"Real" fisheye perspectives.*

with refraction at the cornea. Minor corrections then occur at the lens inside the eye (Figure 20). Thus, the images sent from the retina to our brains are distorted anyway.

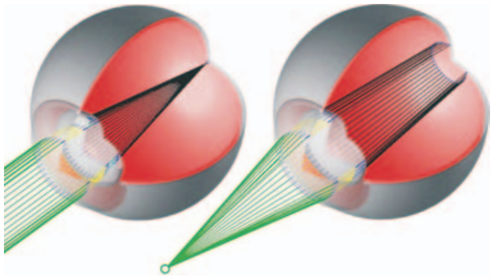

Figure 20: *Refraction at cornea and eye lens.*

6. Reflections as primary perspectives

Reflections in planes produce ordinary (classic) perspectives. The reflected light rays from space points S through the eye point E (Figure 21 left) run through the reflected point S^*, which may be interpreted as part of the reflected (and of course virtual) space.

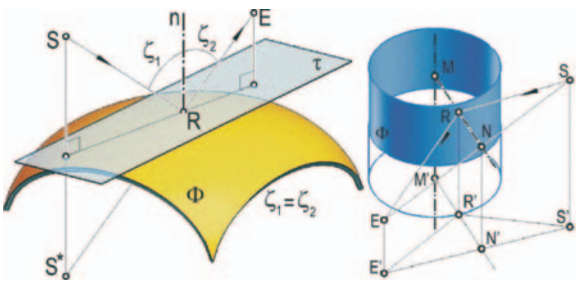

Figure 21: *Reflections in curved surfaces.*

Reflections in any other surface Φ will lead to skewed images, which we can count to nonlinear primary perspectives.

Special cases (reflections in a sphere or in a cylinder of revolution) lead to perspectives like in Figure 21 right and 22. Escher and Dalí, e.g., worked with such "anamorphoses" ([Sec04]).

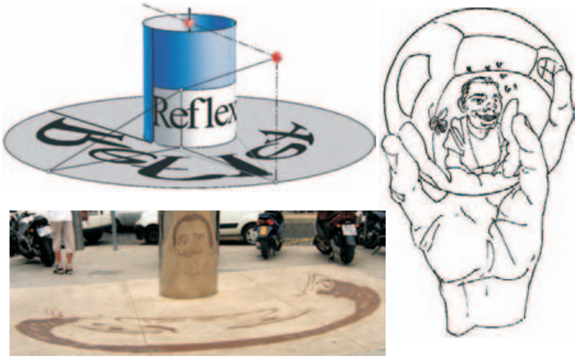

Figure 22: *Reflections in a cylinder and a sphere.*

7. Secondary nonlinear perspectives

So far we have spoken about perspectives that depict space objects into image surfaces – via direct projection, refraction or reflection.

A second group of nonlinear perspectives is generated indirectly from primary perspectives: First, space is projected onto a surface. Second, the points on this surface are either projected or transformed into the drawing plane by means of formulas respectively.

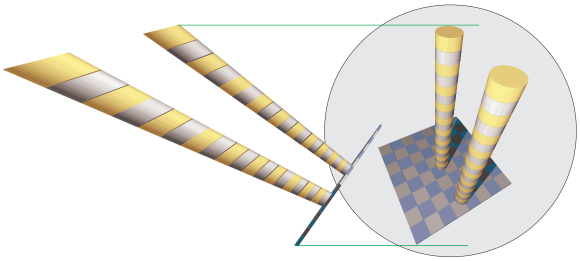

Figure 23: *Linearly transformed "well known objects"*

A very simple example of this is a photo of a photo. Since both transformations are linear, the secondary result is linear as well. However, in general it does not originate from an existing object in space. Reversely, under certain conditions we may fit linearly transformed objects into the secondary viewing rays (Figure 23). This is used in theater architecture or art.

7.1. Projections and other transformations of the sphere

We project space again onto a sphere around the projection center. When we view the spherical image from the center of the sphere, we cannot tell any difference to the situation in space. We could call the projection a *spherically curved perspective*.

In order to get a planar image corresponding to the curved

Figure 24: *The spherical projection of the sphere.*

perspective we now apply a linear projection. Two well known projections of the spherical image lead to remarkable results: The *stereographic projection* from a point on the sphere onto a plane parallel to the tangent plane in the projection center (Figure 24) and the *gnomonic projection* from the center onto any plane (Figure 26).

Figure 25: *Special projection of a spherical perspective.*

The spherical projection creates secondary nonlinear perspectives of space, where the silhouette of a sphere is always a circle (Figure 25). Straight lines in space first convert to small circles on the projection sphere (they run through the projection center) and finally are projected into circles in the drawing plane. Thus, such a secondary perspective is nonlinear. We will later see that it is an idealized special case of fisheye perspectives – and therefore is "almost primary".

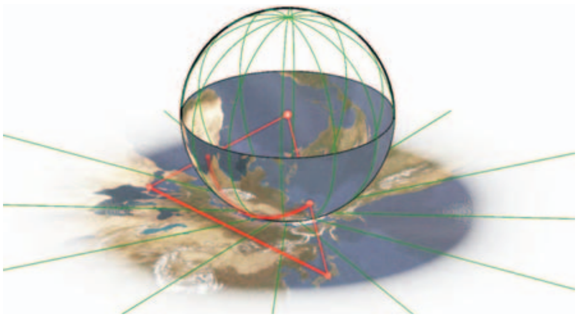

Figure 26: *The gnomonic projection of the sphere.*

As an application, Buckminster Fuller developed a map of the earth by means of projecting the sphere onto several planes – the sides of a cuboctahedron (Figure 27). In the corresponding map, flight routes appear partly as straight lines.

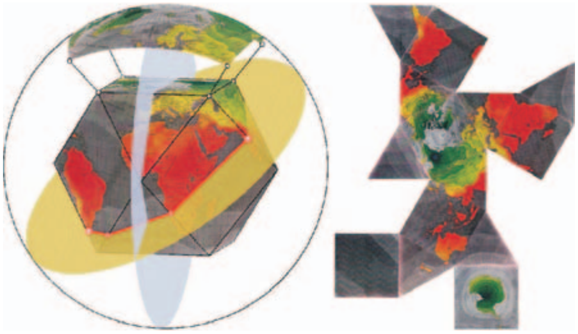

Figure 27: *Gnomonic projections onto a cuboctahedron.*

The perspective produced by gnomonic projection is trivially an ordinary linear perspective and still a primary perspective. It is, however, a useful tool to get maps of the earth, where the shortest distances are straight lines (Figure 26).

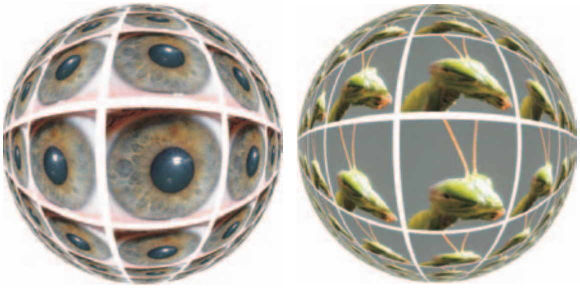

Figure 28: *Normal projection of the sphere.*

As a final special case, one can project the contents of the sphere orthogonally onto a plane. Figure 28 shows such projections for two alleged completely different kinds of eyes that work with a similar principle.

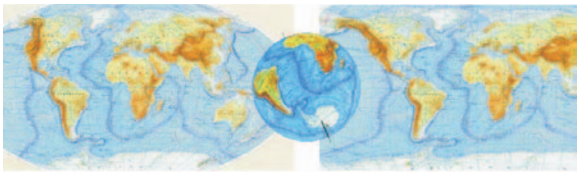

Figure 29: *Winkel's and "simple" model of the sphere*

Among the dozens of mathematical methods to map a sphere into the plane, we mention two: The first is to simply interpret the longitude and the latitude angle as Euclidian coordinates ("cylindrical equidistant mapping", Figure 29 right), the second is named after its inventor O. Winkel (1914) and comes with complicated formulas. The drawback of the simpler mapping is that objects at the edges are *extremely* skewed, whereas Winkel's method leads to optically satisfying results and is therefore often to be found in atlases.

7.2. Models of Non-Euclidean Geometries

When we do not accept the "parallel axiom" in geometry, we get two new kinds of non-Euclidean geometry. For Euclidean-thinking people like us, these geometries are very unusual and hard to imagine. Nevertheless, they play an important role in modern physics. We can only work with *models* of such geometries, and these models are nonlinear, even if they can be interpreted as primary perspectives (ordinary central projections).

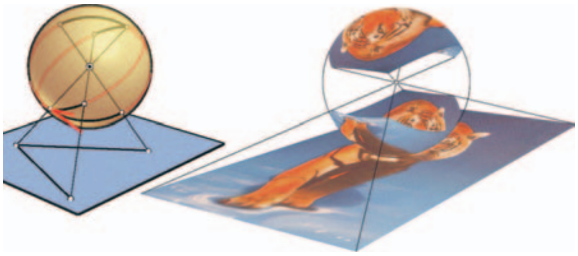

Figure 30: *Elliptic geometry*

Figure 30 shows how the elliptic plane can be projected gnomonically onto a sphere. Then, sphere geometry can be applied as usual (e.g., one can measure angles or distances on the sphere).

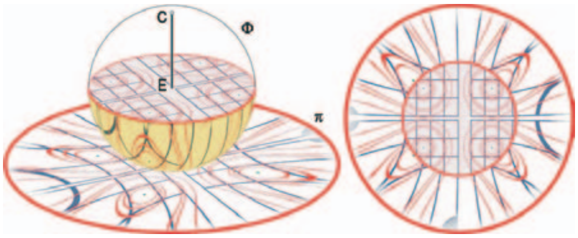

Figure 31: *One model of hyperbolic geometry*

If we project the hyperbolic plane π orthogonally onto the sphere Φ (Figures 31) and then stereographically back onto the plane (center C), we also can deduct and apply some rules of sphere geometry. It is even better to project the hyperbolic plane gnomonically (Figure 32 left) onto a two-sheeted hyperboloid Φ (center E) in order to measure angles and lengths (left).

In any case, straight lines are transformed and therefore the models of non-Euclidean planes always look somehow like nonlinear perspectives. Thus we will classify such models as "nonlinear perspectives in a wider sense". A stereographic projection of the geometry on the hyperboloid (Fig-

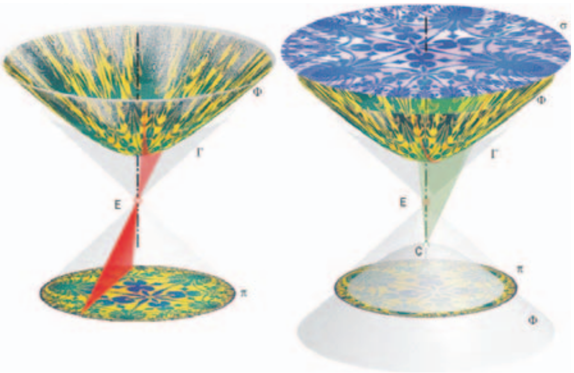

Figure 32: *Hyperbolic geometry, parqueting (J. Wallner)*

ure 32 right, bluish: projection plane σ, projection center C) can even be interpreted as a *primary* perspective of π.

7.3. Inversion

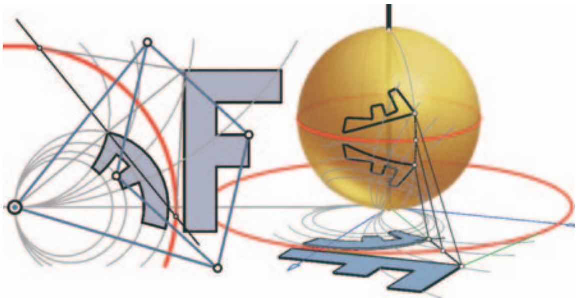

Figure 33: *Inversion in 2-space, spatial interpretation*

Two-dimensional inversion at a circle can be interpreted via a stereographic projection onto a sphere. Straight lines and circles are in general transformed into circles (Figure 33). (This is why we also have circle-models of non-Euclidean geometries.)

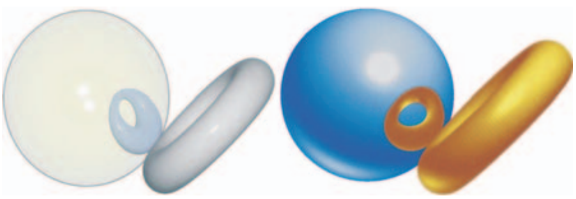

Figure 34: *Inversion at and reflection in a sphere*

Inversions can also be applied in space (inversion at a sphere). Then, planes and spheres are in general transformed into spheres. Figure 34 illustrates, how spatial inversions (in this case of a torus) can look very similar to reflections in spheres, and therefore we also count perspectives of inverted space to the nonlinear perspectives in a wider sense.

8. Realtime conversion of certain nonlinear perspectives

We earlier mentioned that stereographic projections of space projection onto a sphere are idealized special cases of fisheye perspectives. In [GG99] realtime algorithms for rendering scenes in that projection are given. Here we want to "reconstruct" such projections.

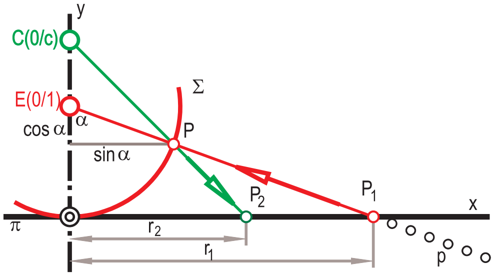

Figure 35: *Projection of the sphere onto the plane*

We need a short calculation in a meridian plane (Figure 35): We identify the optical axis with the y-axis, the image plane $\pi \perp y$ with the x-axis. We let Σ be a unit sphere with center $E(0/1)$ and p be a projection ray with polar angle α. The intersection $P_1 = p \cap \pi$ (the central and primary projection of arbitrary points on p) has radial distance r_1 from the axis. The intersection point $P = p \cap \Sigma$ has the coordinates $P(\sin\alpha / 1 - \cos\alpha)$. Now we choose a center $C(0/c)$ on the optical axis. When we project P from C onto π, we get the secondary projection point P_2. We have

$$r_1 = \tan\alpha \quad \text{and} \quad r_2 : \sin\alpha = c : (c - 1 + \cos\alpha).$$

This leads to a function

$$r_2 = (c \sin \arctan r_1)/(c - 1 + \cos \arctan r_1) = f(r_1)$$

that allows to calculate the secondary radial distance directly from the primary distance via a function f. Reversely, when the secondary projection (r_2) is given, we can convert the corresponding primary perspective by means of the inverse function $r_1 = f^{-1}(r_2) = g(r_2)$ (with some effort the explicit formula can be written down). Since one can proof that f is strictly monotonous growing, we can be sure that g is unique for any value. The converted image is the primary classic perspective.

In the case of stereographic projection, we have $c = 2$. The formula, however, covers any position of C. Idealized fisheye lenses will in fact come close to this formula with values $c > 2$. For large c we have the typical distortions of ultra wide angle lenses at the borders (even expensive lenses have some "cushion effect" which corresponds to large c-values).

Therefore, attempts to undistort cushion effects in ordinary photos and even fisheye photos turned out to be amazingly successful ([ZB95]). Let $\tilde{g} \approx f^{-1}(r_1)$ be a function of the radius that does the job of diminishing lens errors best.

Then the question is: How can we describe $\tilde{g}$? Here Taylor series expansion comes in handy. We always have

$$\tilde{g}(r_1) = \sum_{k=0}^{\infty} a_k r_1^k.$$

Since distortion in the center of the image will be close to zero, we have $\tilde{g}(r_1) = r_1$ for small r_1, i.e., $a_0 = 0$ and $a_1 = 1$ and therefore

$$\tilde{g}(r_1) \approx r + a_2 r_1^2 + a_3 r^3 + a_4 r_1^4 + a_5 r^5 \ldots$$

By interactively choosing values for those coefficients, smaller lens errors can be well eliminated. The result comes very close to classic linear perspective.

Figure 36: *The quadratic grid in a plane helps*

Figure 36 shows how even fisheye perspectives can be undistorted successfully this way in many cases. A "proof" can only be given when one can detect several straight lines in the image that must not coincide with the center (else, they are straight even in the fisheye perspective). In this case, the quadratic grid on the floor was useful. Once the best coefficients are found, they can be used for the calibration of a lens.

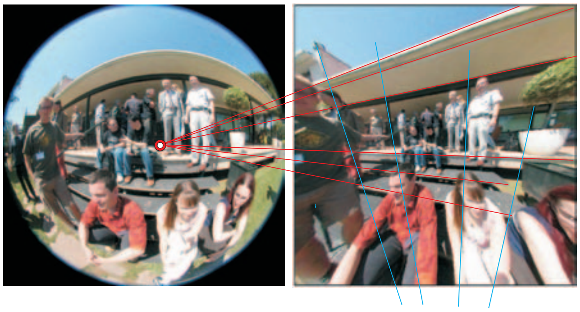

Figure 37: *Vanishing points*

More extreme distortion (Figure 37) may require the coefficients a_6 and a_7 (depending on the focus). Again, it is always helpful to observe vanishing points of parallel lines. For larger distances from the center, the new distance then raises with the powers of 7, i.e., points will disappear from the new image. That goes well together with the fact that fisheye lenses are extremely "contracting", whereas classic perspectives with very small focus tend to "explode" at the borders. In fact, this is a strong argument to use fisheye lenses for extreme views.

The above mentioned inverse function $r_1 = f^{-1}(r_2)$ plays an essential role in the whole conversion process. In order to speed up the algorithms, one should definitely work with tables. Then, for every pixel of the new image, we can look for the corresponding pixel in the skewed image, which lies on the radial ray through the principal point. We can also work with symmetries in order to speed up the calculations. With all optimizations implemented, it is possible to reconstruct new images very quickly (images with resolutions up to a million pixels in realtime).

Nonlinear *secondary* perspectives in general cannot be transformed by means of radial functions. Here we need much more information. Figure 29 shows a typical example: Given the skewed image to the left, how can we "reconstruct" the right image which can then be efficiently be applied for mapping onto the sphere (middle) and/or the creation of "photos from the orbit"?

Winkel gave of course (complicated) formulas $x = x(u, v)$, $y = y(u, v)$ for his transformations, where u and v are longitude and latitude on the sphere. For all the pixels of the new image with corresponding coordinates u, v in a centered Cartesian coordinate system, we now apply the formulas, which leads to a pixel in the skewed image, corresponding to x- and y-values. If we apply the algorithm many times, it is worth creating a table where we store the (x, y)-coordinates for each pair (u, v). (In fact, firmware in our brains probably also works with such precalculated tables in order to "rectify" the spherically curved perspective on the retina.)

Figure 38: *Almost artwork. . .*

Figure 38 shows a scene that cannot be depicted reasonably with any wide-angle lens. The "reconstruction" to the right thus only shows parts of the room, and much information is lost. Since the higher coefficients were a bit too large, straight lines “bend the other way round”. The image reminds somehow of non-Euclidean models.

Figure 39: *Non-Euclidean stays non-Euclidean*

Figure 39 applies the earlier described radial conversion to such a non-Euclidean model. Since a parqueting of the hyperbolic plane is done, one might expect congruent elements for well chosen coefficients. This is, however, impossible, since we deal with completely different metrics: It is not the same task to plaster the Euclidean plane and, e.g., the sphere or a hyperboloid (else such parquetings could easily be created by applying a function to all the pixels of a Euclidean drawing).

9. Conclusion

We call a perspective the result of the projection of 3-space onto a plane or a curved surface. We may distinguish between primary perspectives (where only one projection is necessary) and secondary ones (they originate from primary perspectives by means of another projection or any mathematical mapping). Primary perspectives are the classic perspective projection (it is linear) and projections via refraction or reflection (both are nonlinear).

Curved perspectives are natural – in contrast to “classic perspectives” – and therefore are probably more aesthetic. Extreme wide-angle perspectives can (and should) be “normalized” in real time by means of curved perspectives. This can preferably be done by means of stereographic projection of the corresponding spherical curved perspective or (with similar results) with a fisheye lens.

Reflections in spheres, refractions at planes or spheres and non-Euclidean metrics produce curved perspectives of space that are similar to fisheye photographs. They can be “rectified” in real time by similar algorithms.

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