sampling distance t for fixed theta

In[ ]:=
```
$Assumptions = s > 0 && t > 0 && t < 1 && theta > 0 &&
  theta < Pi / 2 && -1 + 4 t - 4 t^2 + Csc[theta]^2 > 0 && xi > 0 && xi < 1 && T > 0
```

Out[ ]= $s > 0 \&\& t > 0 \&\& t < 1 \&\& \text{theta} > 0 \&\& \text{theta} < \frac{\pi}{2} \&\& -1 + 4\,t - 4\,t^2 + \text{Csc[theta]}^2 > 0 \&\& \text{xi} > 0 \&\& \text{xi} < 1 \&\& T > 0$

first, we take the remaining value of the estimator when only sampling the phase function in solid angle domain. note that this contains the correction factor Sin[theta] for the Jacobian from angle theta to solid angle on the hemisphere. we'll insert all spherical coordinates to express it only wrt s, t, and theta:

```
FullSimplify[Abs[4 rr Sin[hh] / (4 rr ^ 2 Cos[2 * hh] - s ^ 2)] / Sin[theta] //.
  {
   hh → ArcSin[s r / Sqrt[s ^ 2 (t - 1 / 2) ^ 2 + s ^ 2 r ^ 2]],
   rr → Sqrt[s ^ 2 (t - 1 / 2) ^ 2 + s ^ 2 r ^ 2],
   r → Sqrt[1 / (4 Sin[theta] ^ 2) - (1 / 2 - t) ^ 2 ] - Sqrt[1 / (4 Sin[theta] ^ 2) - 1 / 4]
  }]
```

Out[ ]= $$\frac{2\,\text{Csc[theta]}\left(-\text{Cot[theta]} + \sqrt{-1 + 4\,t - 4\,t^2 + \text{Csc[theta]}^2}\right)}{s\,\text{Abs}\left[8\,(-1+t)\,t + 2\,\text{Cot[theta]}\left(-\text{Cot[theta]} + \sqrt{-1 + 4\,t - 4\,t^2 + \text{Csc[theta]}^2}\right)\right]}$$

In[ ]:= $$\text{TrigReduce}\left[\frac{2\,\text{Csc[theta]}\left(-\text{Cot[theta]} + \sqrt{-1 + 4\,t - 4\,t^2 + \text{Csc[theta]}^2}\right)}{s\,\text{Abs}\left[8\,(-1+t)\,t + 2\,\text{Cot[theta]}\left(-\text{Cot[theta]} + \sqrt{-1 + 4\,t - 4\,t^2 + \text{Csc[theta]}^2}\right)\right]}\right]$$

Out[ ]= $$-\frac{2\,\text{Csc[theta]}\left(\text{Cot[theta]} - \sqrt{-1 + 4\,t - 4\,t^2 + \text{Csc[theta]}^2}\right)}{s\,\text{Abs}\left[8\,(-1+t)\,t + 2\,\text{Cot[theta]}\left(-\text{Cot[theta]} + \sqrt{-1 + 4\,t - 4\,t^2 + \text{Csc[theta]}^2}\right)\right]}$$

and as it turns out, the Abs part in the denominator is *always* negative. so remove abs and add a sign before integrating:

In[ ]:= $$\text{FullSimplify}\left[\frac{2\,\text{Csc[theta]}\left(\text{Cot[theta]} - \sqrt{-1 + 4\,t - 4\,t^2 + \text{Csc[theta]}^2}\right)}{s\left(8\,(-1+t)\,t + 2\,\text{Cot[theta]}\left(-\text{Cot[theta]} + \sqrt{-1 + 4\,t - 4\,t^2 + \text{Csc[theta]}^2}\right)\right)}\right]$$

Out[ ]= $$\frac{\text{Csc[theta]}\left(\text{Cot[theta]} - \sqrt{-(1-2\,t)^2 + \text{Csc[theta]}^2}\right)}{s\left(4\,(-1+t)\,t + \text{Cot[theta]}\left(-\text{Cot[theta]} + \sqrt{-(1-2\,t)^2 + \text{Csc[theta]}^2}\right)\right)}$$

In[ ]:= Integrate[$\frac{2\,\text{Csc[theta]}\left(\text{Cot[theta]} - \sqrt{-1+4\,t-4\,t^2+\text{Csc[theta]}^2}\right)}{s\left(8\,(-1+t)\,t+2\,\text{Cot[theta]}\left(-\text{Cot[theta]}+\sqrt{-1+4\,t-4\,t^2+\text{Csc[theta]}^2}\right)\right)}$, t]

Out[ ]= $-\frac{1}{2\,s}\,i\,\text{Csc[theta] Log}\left[-2\,i\,(1-2\,t-\text{Cos[2 theta]}+2\,t\,\text{Cos[2 theta]})\,\text{Csc[theta]} + 4\sqrt{\frac{-1-4\,t+4\,t^2-\text{Cos[2 theta]}+4\,t\,\text{Cos[2 theta]}-4\,t^2\,\text{Cos[2 theta]}}{-1+\text{Cos[2 theta]}}}\,\text{Sin[theta]}\right]$

FullSimplify[ComplexExpand[%2]]

Out[ ]= $\frac{\text{Csc[theta]}\left(\text{Arg}\left[-i+2\,i\,t+\sqrt{-1+4\,t-4\,t^2+\text{Csc[theta]}^2}\right]-i\,\text{Log[4]}\right)}{2\,s}$

%3 /. {t → 0}

Out[ ]= $\frac{\text{Csc[theta]}\left(\text{Arg}\left[-i+\sqrt{-1+\text{Csc[theta]}^2}\right]-i\,\text{Log[4]}\right)}{2\,s}$

FullSimplify[ComplexExpand[%3 - %4]]

Out[ ]= $\frac{\left(-\text{Arg}[-i+\text{Cot[theta]}]+\text{Arg}\left[-i+2\,i\,t+\sqrt{-1+4\,t-4\,t^2+\text{Csc[theta]}^2}\right]\right)\text{Csc[theta]}}{2\,s}$

this looks promising, as if it’s actually real valued. replace the Arg by something with an explicit ArcTan and try to find the inverse:

In[ ]:=

FullSimplify[$\frac{\left(-\text{ArcTan}[-1/\text{Cot[theta]}]+\text{ArcTan}\left[(2\,t-1)\Big/\sqrt{-1+4\,t-4\,t^2+\text{Csc[theta]}^2}\right]\right)\text{Csc[theta]}}{2\,s}$]

Out[ ]= $\frac{\left(\text{theta}+\text{ArcTan}\left[\frac{-1+2\,t}{\sqrt{-(1-2\,t)^2+\text{Csc[theta]}^2}}\right]\right)\text{Csc[theta]}}{2\,s}$

In[ ]:= InverseFunction[Function[t, $\frac{\left(\text{theta}+\text{ArcTan}\left[\frac{-1+2\,t}{\sqrt{-(1-2\,t)^2+\text{Csc[theta]}^2}}\right]\right)\text{Csc[theta]}}{2\,s}$]]

InverseFunction: Inverse functions are being used. Values may be lost for multivalued inverses.

Out[ ]= Function[t, $\left(1+\text{Tan[theta} - 2\,s\,t\,\text{Sin[theta]]}^2 - \sqrt{\text{Csc[theta]}^2\,\text{Tan[theta} - 2\,s\,t\,\text{Sin[theta]]}^2+\text{Csc[theta]}^2\,\text{Tan[theta} - 2\,s\,t\,\text{Sin[theta]]}^4}\right)\Big/ \left(2\,(1+\text{Tan[theta} - 2\,s\,t\,\text{Sin[theta]]}^2)\right)$]

we will see later that the pdf p(t|theta) was not normalised and that the integral was theta/(Sin[theta] s). so we'll divide that out to normalise the pdf or respectively arrive at cdf(1)=1:

In[ ]:= **InverseFunction[Function[t, Sin[theta] $\frac{\left(\text{theta} + \text{ArcTan}\left[\frac{-1+2\,t}{\sqrt{-(1-2\,t)^2+\text{Csc[theta]}^2}}\right]\right)\text{Csc[theta]}}{2\,\text{theta}}$]]**

InverseFunction: Inverse functions are being used. Values may be lost for multivalued inverses.

Out[ ]= Function[t,

$$\frac{1+\text{Tan[(-1+2 t) theta]}^2 - \sqrt{\text{Csc[theta]}^2\,\text{Tan[(-1+2 t) theta]}^2 + \text{Csc[theta]}^2\,\text{Tan[(-1+2 t) theta]}^4}}{2\left(1+\text{Tan[(-1+2 t) theta]}^2\right)}$$
]

In[ ]:= **FullSimplify[**

$$\frac{1+\text{Tan[(-1+2 t) theta]}^2 - \sqrt{\text{Csc[theta]}^2\,\text{Tan[(-1+2 t) theta]}^2 + \text{Csc[theta]}^2\,\text{Tan[(-1+2 t) theta]}^4}}{2\left(1+\text{Tan[(-1+2 t) theta]}^2\right)}$$

]

Out[ ]= $\frac{1}{2}\left(1 - \text{Cos[theta - 2 t theta]}^2\,\sqrt{\text{Csc[theta]}^2\,\text{Sec[theta - 2 t theta]}^2\,\text{Tan[theta - 2 t theta]}^2}\right)$

this result also does not depend on the global scaling factor s any more, which is good.

in the following we plot our cdf P(t | theta). it is not normalised yet, but P(1|theta) = theta/(Sin[theta] s):

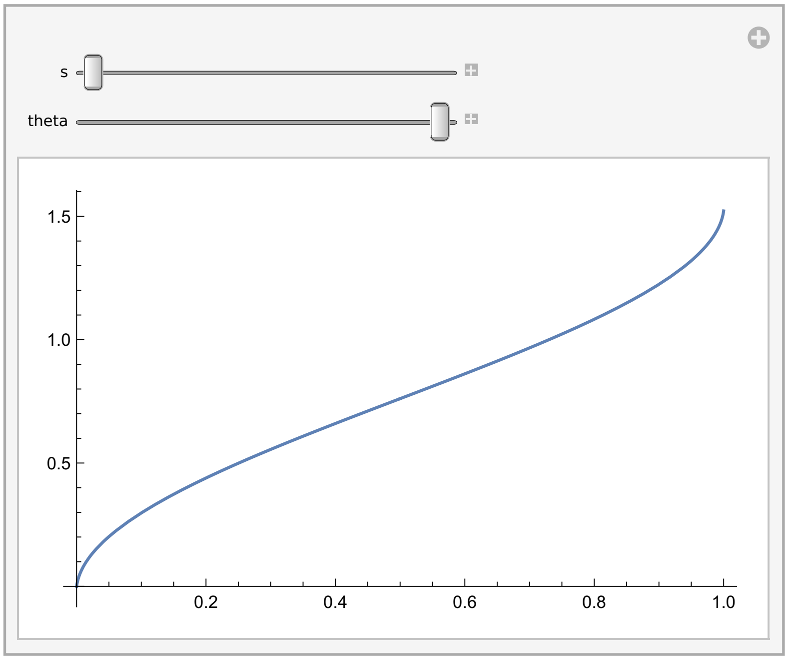
In[ ]:= **FullSimplify[$\frac{\left(\text{theta} + \text{ArcTan}\left[\frac{-1+2\,t}{\sqrt{-(1-2\,t)^2+\text{Csc[theta]}^2}}\right]\right)\text{Csc[theta]}}{2\,s}$ /. t → 1]**

In[ ]:= **FullSimplify[$\frac{\text{(theta + theta) Csc[theta]}}{2\,s}$]**

Out[ ]= $\frac{\text{theta Csc[theta]}}{s}$

In[ ]:=
```
Manipulate[Plot[((theta + ArcTan[(-1+2 t)/Sqrt[-(1-2 t)^2+Csc[theta]^2]]) Csc[theta])/(2 s), {t, 0, 1}],
 {s, 1, 3}, {theta, 0.0001, 1.52}]
```

Out[ ]=

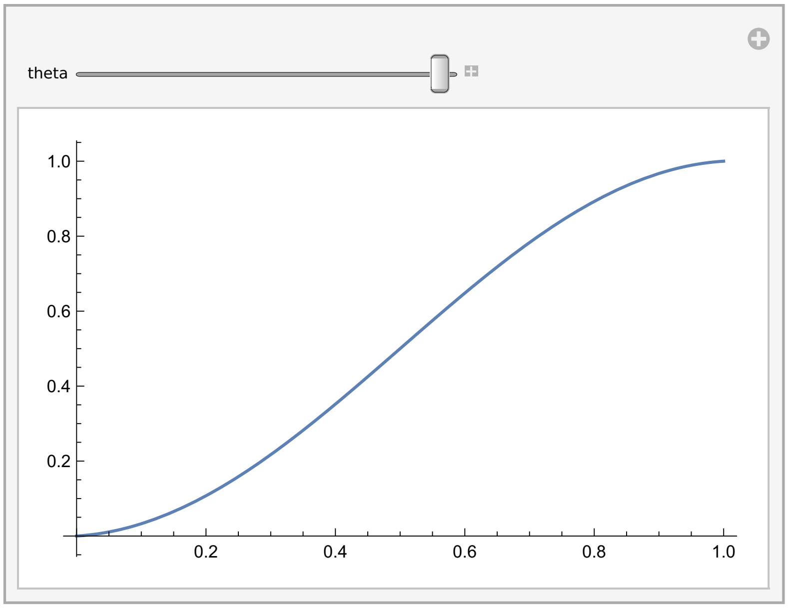


just leaving away the sqrt and the squares gives us the desired inverse:

In[∘]:=

```
Manipulate[
 Plot[1/2 (1 - Cos[theta - 2 t theta]^2 Csc[theta] Sec[theta - 2 t theta] Tan[theta - 2 t theta]),
  {t, 0, 1}], {theta, 0.01, 1.5}]
```

Out[∘]=



this one is the inverse of the normalised cdf and can be used to sample t ~ p(t|theta).