Sampling from Quadric-Based CSG Surfaces

supplemental proofs and derivations

The quadric surface consists of all points that satisfy

$$Q(x) = x^T A x - 2x^T b + c = 0, \tag{1}$$

given a symmetric $A \in \mathbb{R}^{3\times 3}$, $b \in \mathbb{R}^3$, $c \in \mathbb{R}$.

1 Intersection with a Line

Given a line $p + tv$ with $p, v \in \mathbb{R}^3$, $t \in \mathbb{R}$, and $||v|| = 1$, we solve $Q(p + tv) = 0$ for t:

$$\begin{array}{rrcl} & 0 & = & Q(p+tv) \\ \Leftrightarrow & 0 & = & (p+tv)^T A(p+tv) - 2(p+tv)^T b + c \\ \Leftrightarrow & 0 & = & v^T A v^T t^2 + 2(v^T Ap - v^T b)t + (p^T Ap - 2p^T b + c) \quad \mid (*) \\ \Leftrightarrow & 0 & = & t^2 - 2\frac{v^T b - v^T Ap}{v^T A v^T} \cdot t + \frac{p^T Ap - 2p^T b + c}{v^T A v^T} \end{array} \tag{2}$$

The step $(*)$ is only valid if $v^T Av \neq 0$. In that case the solutions are

$$t_{1,2} = \frac{v^T b - v^T Ap}{v^T Av} \pm \sqrt{\left(\frac{v^T b - v^T Ap}{v^T Av}\right)^2 - \frac{p^T Ap - 2p^T b + c}{v^T Av}}. \tag{3}$$

Otherwise, the equation becomes linear and the only solution is

$$t = \frac{1}{2} \cdot \frac{p^T Ap - 2p^T b + c}{v^T b - v^T Ap}. \tag{4}$$

2 Heightfield Function

Consider three orthogonal eigenvectors u_i, u_j, u_k of A (with associated eigenvalues $\lambda_i, \lambda_j, \lambda_k$). As eigenvectors, the following holds:

$$u_i^T A u_i = \lambda_i, \quad u_j^T A u_j = \lambda_j, \quad u_k^T A u_k = \lambda_k \tag{5}$$

If p lies on a quadric sampling plane and d is orthogonal to it, Equation 3 simplifies significantly. p can be represented as $q + xu_i + yu_j$ for quadric center q (satisfying $Aq - b = 0$) and $x, y \in \mathbb{R}$. d is u_k. We simplify Equation 3 in two parts:

$$\begin{aligned}
\tfrac{v^T b - v^T A p}{v^T A v} &= \tfrac{u_k^T b - u_k^T A(q + xu_i + yu_j)}{u_k^T A u_k} && \mid u_k^T A u_k = \lambda_k \\
&= \tfrac{1}{\lambda_k}\left[u_k^T b - u_k^T A(q + xu_i + yu_j)\right] \\
&= \tfrac{1}{\lambda_k}\left[u_k^T(b - Aq) - u_k^T u_i \cdot x + u_k^T u_j \cdot y)\right] && \mid Aq - b = 0 \\
&= \tfrac{1}{\lambda_k}\left[u_k^T u_i \cdot x + u_k^T u_j \cdot y)\right] && \mid u_k^T u_i = 0, \quad u_k^T u_j = 0 \\
&= 0
\end{aligned} \tag{6}$$

Thus, the solutions to the line intersection are:

$$\begin{aligned}
& t_{1,2} \\
= \; & \pm\sqrt{-\tfrac{p^T A p - 2p^T b + c}{v^T A v}} \\
= \; & \pm\sqrt{-\tfrac{(q+xu_i+yu_j)^T A (q+xu_i+yu_j) - 2(q+xu_i+yu_j)^T b + c}{u_k^T A u_k}} \\
= \; & \pm\sqrt{-\tfrac{1}{\lambda_k}\left[(q + xu_i + yu_j)^T A (q + xu_i + yu_j) - 2(q + xu_i + yu_j)^T b + c\right]} \\
= \; & \pm\sqrt{-\tfrac{1}{\lambda_k}\left[x^2 u_i^T A u_i + y^2 u_j^T A u_j - q^T A q + 2q^T A(q + xu_i + yu_j) - 2b^T(q + xu_i + yu_j) + c\right]} \\
= \; & \pm\sqrt{-\tfrac{1}{\lambda_k}\left[\lambda_i x^2 + \lambda_j y^2 - q^T A q + 2(Aq - b)^T(q + xu_i + yu_j) + c\right]} \\
= \; & \pm\sqrt{-\tfrac{1}{\lambda_k}\left[\lambda_i x^2 + \lambda_j y^2 - q^T A q + c\right]} \\
= \; & \pm\sqrt{-\tfrac{1}{\lambda_k}\left[\lambda_i x^2 + \lambda_j y^2 - q^T b + c\right]} \\
= \; & \pm\sqrt{\alpha_x x^2 + \alpha_y y^2 + \alpha_c}
\end{aligned} \tag{7}$$

with $\alpha_x = -\frac{\lambda_i}{\lambda_k}$, $\alpha_y = -\frac{\lambda_j}{\lambda_k}$, and $\alpha_c = \frac{q^T b - c}{\lambda_k}$.

If $\lambda_k = 0$, we start with Equation 4:

$$\begin{aligned}
t &= \tfrac{1}{2} \cdot \tfrac{p^T A p - 2p^T b + c}{v^T b - v^T A p} \\
&= \tfrac{1}{2} \cdot \tfrac{(q+xu_i+yu_j)^T A (q+xu_i+yu_j) - 2(q+xu_i+yu_j)^T b + c}{u_k^T b - u_k^T A (q+xu_i+yu_j)} \\
&= \tfrac{1}{2} \cdot \tfrac{(q+xu_i+yu_j)^T A (q+xu_i+yu_j) - 2(q+xu_i+yu_j)^T b + c}{u_k^T b} && \mid (*) \\
&= \tfrac{1}{2} \cdot \tfrac{\lambda_i x^2 + \lambda_j y^2 - q^T b + c}{u_k^T b} \\
&= \beta_x x^2 + \beta_y y^2 + \beta_c
\end{aligned} \tag{8}$$

with $\beta_x = -\frac{\lambda_i}{2u_k^T b}$, $\beta_y = -\frac{\lambda_j}{2u_k^T b}$, and $\beta_c = -\frac{q^T b - c}{2u_k^T b}$. There is no intersection with the quadric if $u_k^T b = 0$. This second version is mostly used for paraboloid quadrics.

3 Convergence of Conservative Distance Function

For quadrics, we derived the following bounds on the quadric value $Q(x)$, given a reference point x_0:

$$Q(x_0 + t \cdot v) \leq Q(x_0) + t \cdot 2||Ax_0 - b|| + t^2 \cdot \max \lambda_i \tag{9}$$

$$Q(x_0 + t \cdot v) \geq Q(x_0) - t \cdot 2||Ax_0 - b|| + t^2 \cdot \min \lambda_i \tag{10}$$

We use these bounds for a conservative distance-to-quadric-surface by solving for the smallest positive t, where the bound is equal to zero.

Here, we briefly prove that not only the absolute error, but also the relative error goes to zero close to the surface. Consider Equation 9. This equation is used when $Q(x_0)$ is negative, i.e. x_0 is inside the surface. For small t, the quadratic term $t^2 \cdot \max \lambda_i$ can be ignored and only the relative error of $t \cdot 2||Ax_0 - b||$ is important, which is our v-independent approximation of $2 \cdot t \cdot (Ax_0 - b)^T v$. Thus, the relative error (for small t) reduces to

$$\eta = \frac{(Ax_0 - b)^T v - ||Ax_0 - b||}{(Ax_0 - b)^T v}, \tag{11}$$

where v is the direction of the shortest distance to the quadric surface. Close to the surface (and inside of it), v is converging to the direction of the gradient of $Q(x_0)$, which is proportional to $Ax_0 - b$. Therefore $(Ax_0 - b)^T v$ converges to $||Ax_0 - b||$ and thus η to 0.

Outside the surface, Equation 10 is used. Here, the error is between $(Ax_0 - b)^T v$ and $-||Ax_0 - b||$. However, if the surface is reached from outside, v points into negative gradient direction and thus η converges to zero here as well.

4 Heightfield Distortion

Given the heightfield function $h(x, y) = \sqrt{\alpha_x x^2 + \alpha_y y^2 + \alpha_c}$, The surface normal $n(x, y)$ is proportional to $(-\frac{\partial h}{\partial x}, -\frac{\partial h}{\partial y}, 1)^T$, which is

$$\frac{1}{\sqrt{\alpha_x x^2 + \alpha_y y^2 + \alpha_c}} \begin{pmatrix} -\alpha_x x \\ -\alpha_y y \\ 1 \end{pmatrix}. \tag{12}$$

We continue with the following (unnormalized) normal for ease-of-use:

$$n(x, y) = \begin{pmatrix} -\alpha_x x \\ -\alpha_y y \\ \sqrt{\alpha_x x^2 + \alpha_y y^2 + \alpha_c} \end{pmatrix}. \tag{13}$$

The distortion from 2D to 3D can be measured by

$$\delta(x,y) = \left[\left(\frac{n(x,y)}{|n(x,y)|} \right)^T \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right]^{-1}, \tag{14}$$

the inverted projection of the (normalized) normal onto the up vector. This unfolds to

$$\begin{aligned} \delta(x,y) &= \tfrac{|n(x,y)|}{n(x,y)_2} \\ &= \frac{\sqrt{\alpha_x^2 x^2 + \alpha_y^2 y^2 + \sqrt{\alpha_x x^2 + \alpha_y y^2 + \alpha_c}^2}}{\sqrt{\alpha_x x^2 + \alpha_y y^2 + \alpha_c}} \\ &= \sqrt{\tfrac{(\alpha_x + \alpha_x^2) x^2 + (\alpha_y + \alpha_y^2) y^2 + \alpha_c}{\alpha_x x^2 + \alpha_y y^2 + \alpha_c}}. \end{aligned} \tag{15}$$

We are only considering and evaluating the regions where the heightfield is defined, i.e. where $\sqrt{\alpha_x x^2 + \alpha_y y^2 + \alpha_c}$ is real. Thus, $\alpha_x x^2 + \alpha_y y^2 + \alpha_c \geq 0$. As $(\alpha_x + \alpha_x^2)x^2 + (\alpha_y + \alpha_y^2)y^2 + \alpha_c \geq \alpha_x x^2 + \alpha_y y^2 + \alpha_c$ (nominator is at least as large as denominator), we have $\delta(x,y) \geq 1$.

During our adaptive sampling, we require an upper bound on the distortion for a given basic region. $\sqrt{3}$ is a trivial bound, but we can exploit the monotony behavior of $\delta(x,y)$ for better bounds.

The monotony of $\delta(x,y)$ in x direction is classified by the sign of $\frac{\partial}{\partial x}\delta(x,y)$. As $\delta(x,y)$ is continuously differentiable, monotony in x can only change if $\frac{\partial}{\partial x}\delta(x,y) = 0$. This can only happen at a certain line:

$$\begin{aligned} & 0 = \tfrac{\partial}{\partial x}\delta(x,y) \\ \Leftrightarrow\ & 0 = \tfrac{1}{2\delta(x,y)} \left(\tfrac{2(\alpha_x^2 + \alpha_x)x}{\alpha_x x^2 + \alpha_y y^2 + \alpha_c} - \tfrac{2\alpha_x x((\alpha_x + \alpha_x^2)x^2 + (\alpha_y + \alpha_y^2)y^2 + \alpha_c)}{(\alpha_x x^2 + \alpha_y y^2 + \alpha_c)^2} \right) \\ \Leftrightarrow\ & 0 = \tfrac{2(\alpha_x^2 + \alpha_x)x}{\alpha_x x^2 + \alpha_y y^2 + \alpha_c} - \tfrac{2\alpha_x x((\alpha_x + \alpha_x^2)x^2 + (\alpha_y + \alpha_y^2)y^2 + \alpha_c)}{(\alpha_x x^2 + \alpha_y y^2 + \alpha_c)^2} \\ \Leftrightarrow\ & 0 = 2(\alpha_x^2 + \alpha_x)x(\alpha_x x^2 + \alpha_y y^2 + \alpha_c) - 2\alpha_x x((\alpha_x + \alpha_x^2)x^2 + (\alpha_y + \alpha_y^2)y^2 + \alpha_c) \\ \Leftrightarrow\ & 0 = x \cdot \left[(\alpha_x^2 + \alpha_x)(\alpha_x x^2 + \alpha_y y^2 + \alpha_c) - \alpha_x((\alpha_x + \alpha_x^2)x^2 + (\alpha_y + \alpha_y^2)y^2 + \alpha_c) \right] \end{aligned} \tag{16}$$

Thus, the monotony in x can only change at $x = 0$ or if

$$\begin{aligned} & (\alpha_x^2 + \alpha_x)(\alpha_x x^2 + \alpha_y y^2 + \alpha_c) = \alpha_x((\alpha_x + \alpha_x^2)x^2 + (\alpha_y + \alpha_y^2)y^2 + \alpha_c) \\ \Leftrightarrow\ & (\alpha_x^2 + \alpha_x)(\alpha_y y^2 + \alpha_c) = \alpha_x((\alpha_y + \alpha_y^2)y^2 + \alpha_c) \\ \Leftrightarrow\ & (\alpha_x^2 \alpha_y - \alpha_x \alpha_y^2)y^2 = -\alpha_x^2 \alpha_c \\ \Leftrightarrow\ & (\alpha_x \alpha_y - \alpha_y^2)y^2 = -\alpha_x \alpha_c \\ \Leftrightarrow\ & y = \pm\sqrt{\tfrac{\alpha_x \alpha_c}{(\alpha_y - \alpha_x)\alpha_y}}, \end{aligned} \tag{17}$$

should those two lines exist. Analogously, monotony in y can only change at $y = 0$ or if

$$x = \pm\sqrt{\frac{\alpha_y \alpha_c}{(\alpha_x - \alpha_y)\alpha_x}}. \tag{18}$$

Note that if one of these pairs of lines exist (and is different from the coordinate axes), the other does not, because

$$\begin{aligned} & \tfrac{\alpha_x \alpha_c}{(\alpha_y - \alpha_x)\alpha_y} \cdot \tfrac{\alpha_y \alpha_c}{(\alpha_x - \alpha_y)\alpha_x} \\ = \; & -\tfrac{\alpha_c^2}{(\alpha_y - \alpha_x)^2} \\ < \; & 0. \end{aligned} \tag{19}$$

Thus, the terms under the square root have opposite signs: if one is positive, the other is negative. The case than any α is zero can be ignored, because the pair of lines would be collapse to a coordinate axis.

Our basic regions are fully contained in a quadrant. When restricted to a single quadrant, we have proven that $\delta(x, y)$ consists of two regions where it is monotone in x and y. These two regions are either separated by a horizontal or a vertical line and only a single monotony direction changes across the line.

Now consider an axis-aligned rectangle R. If R is fully contained in either monotony region, it is trivial to see that the extrema are assumed at a corner of R: Inside R, $\delta(x, y)$ is monotone in x and y. If R contains both regions, the situation is a bit more complex. If the regions are separated by a vertical line, then the monotony of y changes from left to right. However, x is monotone in the complete quadrant and the extrema in R cannot lie on the vertical line but must be on the left or right edge of R. Similarly, if the change of monotony is along a horizontal line, the complete quadrant is monotony in y and extrema must be on the upper or lower edge of R. Thus, while $\delta(x, y)$ might not be completely monotone in R, we still have the useful property that the extrema of $\delta(x, y)$, i.e. the minimum and maximum distortion, are assumed at a corner of R and can thus be computed by 4 evaluations of $\delta(x, y)$.

5 Parabolic Cases

In the parabolic cases, the heightfield is $h(x, y) = \beta_x x^2 + \beta_y y^2 + \beta_c$. The (unnormalized) normal is:

$$n(x, y) = \begin{pmatrix} -2\beta_x x \\ -2\beta_y y \\ 1 \end{pmatrix}. \tag{20}$$

The distortion then evaluates to

$$\begin{aligned} \delta(x, y) &= \tfrac{|n(x,y)|}{n(x,y)_2} \\ &= \sqrt{4\beta_x x^2 + 4\beta_y y^2 + 1}. \end{aligned} \tag{21}$$

Inside any given quadrant, this function is monotone in x and y, thus also satisfying our distortion-extrema-lie-on-AABB-corner property.

For the decomposition into low-distortion regions, we solve $|n(x,y)_2| = |n(x,y)_0|$ and $|n(x,y)_2| = |n(x,y)_1|$, which yield the lines

$$x = \pm\frac{1}{2\beta_x} \quad \text{and} \quad y = \pm\frac{1}{2\beta_y}. \tag{22}$$

6 Splat Box Height

When rendering the splats as surface-aligned boxes, we can use the distortion bounds to compute the box thickness. The distortion δ is the local slope of the heightfield, the extrema the local worst-case slopes (δ^+ and δ^-). Given a point p and radius r, we want to compute a bound of the maximum deviation of the heightfield from the splat plane within radius r. In 2D (cut across the highest curvature direction), the splat plane (which is now a line) has the following direction vector:

$$d = \frac{1}{\sqrt{1+\delta^2}}\begin{pmatrix}1\\ \delta\end{pmatrix} \tag{23}$$

Similarly, the maximum distortion gives rise to the following direction:

$$d^+ = \frac{1}{\sqrt{1+\delta^{+2}}}\begin{pmatrix}1\\ \delta^+\end{pmatrix} \tag{24}$$

A bound on the half-extent of the splat box in positive normal direction is how far the lines of d and d^+ have diverged when measured in normal direction at distance r from p. This can be computed as $r \cdot \sin\alpha$, where α is the angle between d and d^+:

$$\begin{aligned}
& r \cdot \sin\alpha \\
= \; & r \cdot \sqrt{1-\cos^2\alpha} \\
= \; & r \cdot \sqrt{1-(d^T d^+)^2} \\
= \; & r \cdot \sqrt{1-\left(\tfrac{1+\delta\delta^+}{\sqrt{(1+\delta^2)(1+\delta^{+2})}}\right)^2} \\
= \; & r \cdot \sqrt{1-\tfrac{(1+\delta\delta^+)^2}{(1+\delta^2)(1+\delta^{+2})}} \\
= \; & r \cdot \sqrt{\tfrac{(1+\delta^2)(1+\delta^{+2})-(1+\delta\delta^+)^2}{(1+\delta^2)(1+\delta^{+2})}} \\
= \; & r \cdot \sqrt{\tfrac{(\delta^+-\delta)^2}{(1+\delta^2)(1+\delta^{+2})}} \\
= \; & r \cdot (\delta^+-\delta) \cdot \tfrac{1}{\sqrt{(1+\delta^2)(1+\delta^{+2})}}
\end{aligned} \tag{25}$$

Similarly, in the other direction we obtain

$$r \cdot (\delta-\delta^-) \cdot \frac{1}{\sqrt{(1+\delta^2)(1+(\delta^-)^2)}} \tag{26}$$

In our implementation, we dropped the $\frac{1}{\sqrt{(1+\delta^2)(1+\delta^{+2})}}$ factor, which is smaller than 1. We only need an upper bound and $r \cdot (\delta^+ - \delta^-)$ was already small enough for most splats that we could render a single quad instead of a box.