Tech Document

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This tech document is organized into four sections. Section 1 covers in greater detail the derivation of our pressure projection from the incompressible Euler equations. Section 2 discusses implementation details for the matrices derived in Section 1, including expressions for the entries in terms of local cell indices. Section 3 goes into detail about our cut cell formulation and the necessary modifications to the various matrices. Finally, Section 4 shows that standing pool is a solution of our discretized system.

1 Pressure Projection

After splitting, the weak forms of the incompressible Euler equations are

$$\int_{\Omega} \mathbf{r} \cdot \rho \left(\frac{\mathbf{u}^{n+1} - \mathbf{w}}{\Delta t} \right) d\mathbf{x} = \int_{\Omega} p^{n+1} \nabla \cdot \mathbf{r} + \rho \mathbf{r} \cdot \mathbf{g} d\mathbf{x} - \int_{\partial\Omega} p^{n+1} \mathbf{r} \cdot \mathbf{n} ds(\mathbf{x}), \tag{1}$$

$$\int_{\Omega} q \nabla \cdot \mathbf{u}^{n+1} d\mathbf{x} = 0 \tag{2}$$

with boundary condition

$$\int_{\partial\Omega_D} \mu \left(\mathbf{u}^{n+1} \cdot \mathbf{n} - a \right) ds(\mathbf{x}) = 0. \tag{3}$$

On the boundary $\partial\Omega_N$ we have $p = 0$, and on the boundary $\partial\Omega_D$ we have the Lagrange multiplier $p^{n+1} = \lambda^{n+1}$. We can then rewrite (1) as

$$\int_{\Omega} \mathbf{r} \cdot \rho \left(\frac{\mathbf{u}^{n+1} - \mathbf{w}}{\Delta t} \right) d\mathbf{x} = \int_{\Omega} p^{n+1} \nabla \cdot \mathbf{r} + \rho \mathbf{r} \cdot \mathbf{g} d\mathbf{x} - \int_{\partial\Omega_D} \lambda^{n+1} \mathbf{r} \cdot \mathbf{n} ds(\mathbf{x}). \tag{4}$$

Let $N_{\mathbf{i}}$ be the multiquadratic B-spline basis function associated with cell center $\mathbf{x}_{\mathbf{i}}$, and let $\chi_{\mathbf{c}}$ be the multilinear B-spline basis function associated with grid node $x_{\mathbf{c}}$.

We interpolate $\mathbf{u}^{n+1}$, $\mathbf{w}$, $\mathbf{r}$, p^{n+1}, λ^{n+1}, q, and μ using these functions as follows:

$$u_{\alpha}^{n+1} = \bar{u}_{\alpha\mathbf{i}}^{n+1} N_{\mathbf{i}},$$

$$
\begin{aligned}
w_\alpha &= \bar{w}_{\alpha\mathbf{i}} N_\mathbf{i} \\
r_\alpha &= \bar{r}_{\alpha\mathbf{i}} N_\mathbf{i} \\
p^{n+1} &= p_\mathbf{c}^{n+1} \chi_\mathbf{c}, \\
\lambda^{n+1} &= \lambda_\mathbf{b}^{n+1} \chi_\mathbf{b}, \\
q &= q_\mathbf{c} \chi_\mathbf{c}, \\
\mu &= \mu_\mathbf{b} \chi_\mathbf{b},
\end{aligned}
$$

where Greek subscripts (α, β, etc.) denote vector components and the subscript $\mathbf{b}$ in place of $\mathbf{c}$ indicates that the grid node is a boundary node. Also note that we have used summation notation.

Substituting these interpolations into the weak form above, we obtain the equations

$$
\begin{aligned}
\int_\Omega \bar{r}_{\alpha\mathbf{i}} N_\mathbf{i} \rho \frac{\bar{u}_{\alpha\mathbf{j}}^{n+1} - \bar{w}_{\alpha\mathbf{j}}}{\Delta t} N_\mathbf{j} d\mathbf{x} = &\int_\Omega p_\mathbf{c}^{n+1} \chi_\mathbf{c} \bar{r}_{\alpha\mathbf{i}} \frac{\partial N_\mathbf{i}}{\partial x_\alpha} d\mathbf{x} \\
&+ \int_\Omega \rho \bar{r}_{\alpha\mathbf{i}} N_\mathbf{i} g_\alpha d\mathbf{x} \\
&- \int_{\partial\Omega_D} \lambda_\mathbf{b}^{n+1} \chi_\mathbf{b} \bar{r}_{\alpha\mathbf{i}} N_\mathbf{i} n_\alpha ds(\mathbf{x}),
\end{aligned} \tag{5}
$$

$$
\int_\Omega q_\mathbf{c} \chi_\mathbf{c} \bar{u}_{\alpha\mathbf{i}}^{n+1} \frac{\partial N_\mathbf{i}}{\partial x_\alpha} d\mathbf{x} = 0, \tag{6}
$$

$$
\int_{\partial\Omega_D} \mu_\mathbf{b} \chi_\mathbf{b} \left(\bar{u}_{\alpha\mathbf{i}}^{n+1} N_\mathbf{i} n_\alpha - a \right) ds(\mathbf{x}) = 0. \tag{7}
$$

Rearranging the terms and using the Kronecker delta function $\delta_{\alpha\beta}$, we can rewrite these three equations as

$$
\begin{aligned}
\bar{r}_{\alpha\mathbf{i}} \left(\delta_{\alpha\beta} \int_\Omega \frac{\rho}{\Delta t} N_\mathbf{i} N_\mathbf{j} d\mathbf{x} \right) \left(\bar{u}_{\beta\mathbf{j}}^{n+1} - \bar{w}_{\beta\mathbf{j}} \right) = &\; \bar{r}_{\alpha\mathbf{i}} \left(\int_\Omega \chi_\mathbf{c} \frac{\partial N_\mathbf{i}}{\partial x_\alpha} d\mathbf{x} \right) p_\mathbf{c}^{n+1} \\
&+ \bar{r}_{\alpha\mathbf{i}} \left(\int_\Omega \rho g_\alpha N_\mathbf{i} d\mathbf{x} \right) \\
&- \bar{r}_{\alpha\mathbf{i}} \left(\int_{\partial\Omega_D} n_\alpha \chi_\mathbf{b} N_\mathbf{i} ds(\mathbf{x}) \right) \lambda_\mathbf{b}^{n+1},
\end{aligned} \tag{8}
$$

$$
q_\mathbf{c} \left(\int_\Omega \chi_\mathbf{c} \frac{\partial N_\mathbf{i}}{\partial x_\alpha} d\mathbf{x} \right) \bar{u}_{\alpha\mathbf{i}}^{n+1} = 0, \tag{9}
$$

$$
\mu_\mathbf{b} \left(\int_{\partial\Omega_D} n_\alpha \chi_\mathbf{b} N_\mathbf{i} ds(\mathbf{x}) \right) \bar{u}_{\alpha\mathbf{i}}^{n+1} = \mu_\mathbf{b} \left(\int_{\partial\Omega_D} a \chi_\mathbf{b} ds(\mathbf{x}) \right). \tag{10}
$$

Since $\bar{r}_{\alpha\mathbf{i}}$, $q_{\mathbf{c}}$, and $\mu_{\mathbf{b}}$ are arbitrary, we can eliminate them from the equations. Next, we define the vectors $\mathbf{U}^{n+1}$, $\mathbf{W}$, $\mathbf{P}^{n+1}$, and $\mathbf{\Lambda}^{n+1}$ to be the vectors with entries $U_{\alpha\mathbf{i}}^{n+1} = \bar{u}_{\alpha\mathbf{i}}^{n+1}$,$W_{\alpha\mathbf{i}} = \bar{w}_{\alpha\mathbf{i}}$, $P_{\mathbf{c}}^{n+1} = p_{\mathbf{c}}^{n+1}$, and $\Lambda_{\mathbf{b}}^{n+1} = \lambda_{\mathbf{b}}^{n+1}$. In other words, they are the vectors containing all $\bar{\mathbf{u}}_{\mathbf{i}}^{n+1}$, $\bar{\mathbf{w}}_{\mathbf{i}}$, $p_{\mathbf{c}}^{n+1}$, and $\lambda_{\mathbf{b}}^{n+1}$, respectively.

Furthermore, we define the following matrices:

$$M_{\alpha\mathbf{i}\beta\mathbf{j}} = \delta_{\alpha\beta} \int_{\Omega} \frac{\rho}{\Delta t} N_{\mathbf{i}} N_{\mathbf{j}} d\mathbf{x},$$

$$D_{\mathbf{c}\alpha\mathbf{i}} = \int_{\Omega} \chi_{\mathbf{c}} \frac{\partial N_{\mathbf{i}}}{\partial x_{\alpha}} d\mathbf{x},$$

$$B_{\mathbf{b}\alpha\mathbf{i}} = \int_{\partial\Omega_D} n_{\alpha} \chi_{\mathbf{b}} N_{\mathbf{i}} ds(\mathbf{x}),$$

and vectors:

$$\hat{g}_{\alpha\mathbf{i}} = \int_{\Omega} \rho g_{\alpha} N_{\mathbf{i}} d\mathbf{x},$$

$$A_{\mathbf{b}} = \int_{\partial\Omega_D} a \chi_{\mathbf{b}} ds(\mathbf{x}).$$

With these definitions, the above equations can be rewritten in the form

$$\mathbf{M}(\mathbf{U}^{n+1} - \mathbf{W}) = \mathbf{D}^T \mathbf{P}^{n+1} - \mathbf{B}^T \mathbf{\Lambda}^{n+1} + \hat{\mathbf{g}}, \tag{11}$$

$$\mathbf{D}\mathbf{U}^{n+1} = \mathbf{0}, \tag{12}$$

$$\mathbf{B}\mathbf{U}^{n+1} = \mathbf{A}. \tag{13}$$

These equations can be written as a single linear system

$$\begin{pmatrix} \mathbf{M} & -\mathbf{D}^T & \mathbf{B}^T \\ -\mathbf{D} & & \\ \mathbf{B} & & \end{pmatrix} \begin{pmatrix} \mathbf{U}^{n+1} \\ \mathbf{P}^{n+1} \\ \mathbf{\Lambda}^{n+1} \end{pmatrix} = \begin{pmatrix} \mathbf{M}\mathbf{W} + \hat{\mathbf{g}} \\ \mathbf{0} \\ \mathbf{A} \end{pmatrix}. \tag{14}$$

Now define the gradient matrix

$$\mathbf{G} = \left[-\mathbf{D}^T, \mathbf{B}^T \right].$$

Then equation (11) becomes

$$\mathbf{M}\mathbf{U}^{n+1} + \mathbf{G} \begin{pmatrix} \mathbf{P}^{n+1} \\ \mathbf{\Lambda}^{n+1} \end{pmatrix} = \mathbf{M}\mathbf{W} + \hat{\mathbf{g}}. \tag{15}$$

Multiplying by $\mathbf{G}^T \mathbf{M}^{-1}$ yields

$$\begin{pmatrix} \mathbf{0} \\ \mathbf{A} \end{pmatrix} + \mathbf{G}^T \mathbf{M}^{-1} \mathbf{G} \begin{pmatrix} \mathbf{P}^{n+1} \\ \mathbf{\Lambda}^{n+1} \end{pmatrix} = \mathbf{G}^T \left(\mathbf{W} - \mathbf{M}^{-1} \hat{\mathbf{g}} \right), \tag{16}$$

where we have applied equations (12) and (13). By inverting the symmetric positive definite matrix $\mathbf{G}^T\mathbf{M}^{-1}\mathbf{G}$, we obtain an expression for $\mathbf{P}^{n+1}$ and $\mathbf{\Lambda}^{n+1}$:

$$\begin{pmatrix} \mathbf{P}^{n+1} \\ \mathbf{\Lambda}^{n+1} \end{pmatrix} = \left(\mathbf{G}^T\mathbf{M}^{-1}\mathbf{G}\right)^{-1} \left(\mathbf{G}^T\left(\mathbf{W} - \mathbf{M}^{-1}\hat{\mathbf{g}}\right) - \begin{pmatrix} \mathbf{0} \\ \mathbf{A} \end{pmatrix}\right). \tag{17}$$

Solving for $\mathbf{U}^{n+1}$ via equation (11), we obtain the velocity correction

$$\mathbf{U}^{n+1} = -\mathbf{M}^{-1}\mathbf{G}\begin{pmatrix} \mathbf{P}^{n+1} \\ \mathbf{\Lambda}^{n+1} \end{pmatrix} + \mathbf{W} + \mathbf{M}^{-1}\hat{\mathbf{g}}. \tag{18}$$

2 Element Matrices

Let Ω^e denote a voxel in the domain. Then we have element analogs of $\mathbf{M}$, $\mathbf{D}$ and $\mathbf{B}$:

$$M^e_{\alpha\mathbf{i}\beta\mathbf{j}} = \delta_{\alpha\beta}\int_{\Omega^e} \frac{\rho}{\Delta t} N_{\mathbf{i}} N_{\mathbf{j}} d\mathbf{x},$$

$$D^e_{\mathbf{d}\alpha\mathbf{i}} = \int_{\Omega^e} \chi_{\mathbf{d}} \frac{\partial N_{\mathbf{i}}}{\partial x_\alpha} d\mathbf{x},$$

$$B^e_{\mathbf{b}\alpha\mathbf{i}} = \int_{\partial\Omega^e_D} n_\alpha \chi_{\mathbf{b}} N_{\mathbf{i}} ds(\mathbf{x}).$$

On the element Ω_e, there are only 4 grid node indices $\mathbf{d}$ and 9 cell center indices $\mathbf{i}$ in 2D for which the corresponding functions $\chi_{\mathbf{d}}$ and $N_{\mathbf{i}}$ are nonzero. Hence, we consider only these indices which are local to this element. In 3D, the corresponding counts are 8 and 27, respectively.

2.0.1 Local to Global Mapping

Consider the mapping

$$\boldsymbol{\phi}_{Ve}(\boldsymbol{\eta}) = \mathbf{x}^e + \Delta x \boldsymbol{\eta}$$

from $[-1/2, 1/2]^2$ to Ω^e, where $\mathbf{x}^e$ is the cell center. For 3D, we map from $[-1/2, 1/2]^3$ to Ω^e.

2.0.2 Local Indexing

Let $\tilde{\mathbf{d}} = (m, n)$ denote the local grid indices corresponding to $\mathbf{d}$ with $m, n \in \{0, 1\}$. Let (i_p, j_p) be the global index of the lower left grid node for the cell. Then the local grid node indices are related to the global grid node indices as follows:

$$\tilde{\mathbf{d}} = (m, n) \to (i_p + m, j_p + n) = \mathbf{d}.$$

Likewise, let $\tilde{\mathbf{i}} = (i, j)$ denote the local cell center indices corresponding to $\mathbf{i}$ with $i, j \in \{-1, 0, 1\}$. Let (i_c, j_c) be the global index of the cell center. Then the local cell center indices are related to the global cell center indices in a similar manner:

$$\tilde{\mathbf{i}} = (i, j) \rightarrow (i_c + i, j_c + j) = \mathbf{i}.$$

For 3D, we use $\tilde{\mathbf{d}} = (l, m, n)$ and $\tilde{\mathbf{i}} = (i, j, k)$. Note that now l corresponds to the first coordinate, and not m.

2.0.3 Local Spline Functions

We also define the local functions

$$\tilde{\chi}_{\tilde{\mathbf{d}}}(\boldsymbol{\eta}) = \tilde{\chi}_m(\eta_1)\tilde{\chi}_n(\eta_2)$$

and

$$\tilde{N}_{\tilde{\mathbf{i}}}(\boldsymbol{\eta}) = \tilde{N}_i(\eta_1)\tilde{N}_j(\eta_2),$$

with the 1D functions defined as follows:

$$\tilde{N}_i(\eta) = \begin{cases} \frac{\left(\frac{1}{2}-\eta\right)^2}{2}, & i = -1 \\ \frac{3}{4} - \eta^2, & i = 0 \\ \frac{\left(\frac{1}{2}+\eta\right)^2}{2}, & i = 1 \end{cases},$$

$$\tilde{\chi}_m(\eta) = \begin{cases} \frac{1}{2} - \eta, & m = 0 \\ \frac{1}{2} + \eta, & m = 1 \end{cases}.$$

Then, we have

$$\chi_{\mathbf{d}}(\mathbf{x}) = \tilde{\chi}_{\tilde{\mathbf{d}}}\left(\boldsymbol{\phi}_{Ve}^{-1}(\mathbf{x})\right) = \tilde{\chi}_{\tilde{\mathbf{d}}}\left(\boldsymbol{\eta}\right)$$

and

$$N_{\mathbf{i}}(\mathbf{x}) = \tilde{N}_{\tilde{\mathbf{i}}}\left(\boldsymbol{\phi}_{Ve}^{-1}(\mathbf{x})\right) = \tilde{N}_{\tilde{\mathbf{i}}}\left(\boldsymbol{\eta}\right).$$

Using the chain rule, we also have

$$\begin{aligned}
\frac{\partial N_{\mathbf{i}}}{\partial x_\alpha}(\mathbf{x}) &= \frac{\partial \tilde{N}_{\tilde{\mathbf{i}}}}{\partial \eta_\beta}\left(\boldsymbol{\phi}_{Ve}^{-1}(\mathbf{x})\right)\frac{\partial \phi_{Ve\beta}^{-1}}{\partial x_\alpha}(\mathbf{x}) \\
&= \frac{\partial \tilde{N}_{\tilde{\mathbf{i}}}}{\partial \eta_\beta}\left(\boldsymbol{\phi}_{Ve}^{-1}(\mathbf{x})\right)\frac{1}{\Delta x}\delta_{\alpha\beta} \\
&= \frac{1}{\Delta x}\frac{\partial \tilde{N}_{\tilde{\mathbf{i}}}}{\partial \eta_\alpha}\left(\boldsymbol{\phi}_{Ve}^{-1}(\mathbf{x})\right) \\
&= \frac{1}{\Delta x}\frac{\partial \tilde{N}_{\tilde{\mathbf{i}}}}{\partial \eta_\alpha}\left(\boldsymbol{\eta}\right).
\end{aligned}$$

2.1 Element Divergence Matrix

We use this change of variable to obtain

$$\begin{aligned} D^e_{\mathbf{d}\alpha\mathbf{i}} &= \int_{\Omega^e} \chi_{\mathbf{d}} \frac{\partial N_{\mathbf{i}}}{\partial x_\alpha} d\mathbf{x} \\ &= \frac{\Delta x^2}{\Delta x} \int_{[-1/2,1/2]^2} \tilde{\chi}_{\tilde{\mathbf{d}}} \frac{\partial \tilde{N}_{\tilde{\mathbf{i}}}}{\partial \eta_\alpha} d\boldsymbol{\eta} \\ &= \Delta x \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \tilde{\chi}_m(\eta_1) \tilde{\chi}_n(\eta_2) \frac{\partial}{\partial \eta_\alpha} \left(\tilde{N}_i(\eta_1) \tilde{N}_j(\eta_2) \right) d\boldsymbol{\eta}. \end{aligned}$$

In the case $\alpha = 1$, we have

$$D^e_{\mathbf{d}1\mathbf{i}} = \Delta x \left(\int_{-1/2}^{1/2} \tilde{\chi}_m(\eta_1) \frac{\partial \tilde{N}_i}{\partial \eta_\alpha}(\eta_1) d\eta_1 \right) \left(\int_{-1/2}^{1/2} \tilde{\chi}_n(\eta_2) \tilde{N}_j(\eta_2) d\eta_2 \right).$$

The first integral is

$$\int_{-1/2}^{1/2} \tilde{\chi}_m(\eta_1) \frac{\partial \tilde{N}_i}{\partial \eta_\alpha}(\eta_1) d\eta_1 = \frac{i^2(6m-3) + 3i - 4m + 2}{12},$$

and the second integral is

$$\int_{-1/2}^{1/2} \tilde{\chi}_n(\eta_2) \tilde{N}_j(\eta_2) d\eta_2 = \frac{-6j^2 + j(2n-1) + 8}{24}.$$

Hence,

$$D^e_{\mathbf{d}1\mathbf{i}} = \Delta x \frac{i^2(6m-3) + 3i - 4m + 2}{12} \frac{(-6)j^2 + j(2n-1) + 8}{24}.$$

For $\alpha = 2$ we have the corresponding equation

$$D^e_{\mathbf{d}2\mathbf{i}} = \Delta x \frac{(-6)i^2 + i(2m-1) + 8}{24} \frac{j^2(6n-3) + 3j - 4n + 2}{12}.$$

In 3D, the analogous equations are

$$D^e_{\mathbf{d}1\mathbf{i}} = \Delta x^2 \frac{i^2(6l-3) + 3i - 4l + 2}{12} \frac{(-6)j^2 + j(2m-1) + 8}{24} \frac{(-6)k^2 + k(2n-1) + 8}{24},$$

$$D^e_{\mathbf{d}2\mathbf{i}} = \Delta x^2 \frac{(-6)i^2 + i(2l-1) + 8}{24} \frac{j^2(6m-3) + 3j - 4m + 2}{12} \frac{(-6)k^2 + k(2n-1) + 8}{24},$$

$$D^e_{\mathbf{d}3\mathbf{i}} = \Delta x^2 \frac{(-6)i^2 + i(2l-1) + 8}{24} \frac{(-6)j^2 + j(2m-1) + 8}{24} \frac{k^2(6n-3) + 3k - 4n + 2}{12}.$$

2.2 Element Mass Matrix

Using the above change of variable as above, we have the following expression for $M^e_{\alpha\mathbf{i}\beta\mathbf{j}}$, with $\tilde{\mathbf{i}} = (i_1, i_2)$ and $\tilde{\mathbf{j}} = (j_1, j_2)$:

$$\begin{aligned}
M^e_{\alpha\mathbf{i}\beta\mathbf{j}} &= \delta_{\alpha\beta} \int_{\Omega^e} \frac{\rho}{\Delta t} N_{\mathbf{i}} N_{\mathbf{j}} d\mathbf{x} \\
&= \delta_{\alpha\beta} \frac{\Delta x^2}{\Delta t} \int_{[-1/2,1/2]^2} \rho \tilde{N}_{\tilde{\mathbf{i}}} \tilde{N}_{\tilde{\mathbf{j}}} d\boldsymbol{\eta} \\
&= \delta_{\alpha\beta} \frac{\Delta x^2}{\Delta t} \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \rho \tilde{N}_{i_1}(\eta_1) \tilde{N}_{i_2}(\eta_2) \tilde{N}_{j_1}(\eta_1) \tilde{N}_{j_2}(\eta_2) d\boldsymbol{\eta}.
\end{aligned}$$

In the case where ρ is constant, we can write this as a product of integrals:

$$M^e_{\alpha\mathbf{i}\beta\mathbf{j}} = \delta_{\alpha\beta} \frac{\rho \Delta x^2}{\Delta t} \left(\int_{-1/2}^{1/2} \tilde{N}_{i_1}(\eta_1) \tilde{N}_{j_1}(\eta_1) d\eta_1 \right) \left(\int_{-1/2}^{1/2} \tilde{N}_{i_2}(\eta_2) \tilde{N}_{j_2}(\eta_2) d\eta_2 \right).$$

The analytic expression for the first integral is

$$\int_{-1/2}^{1/2} \tilde{N}_{i_1}(\eta_1) \tilde{N}_{j_1}(\eta_1) d\eta_1 = \frac{167 i_1^2 j_1^2 - 134(i_1^2 + j_1^2) + 5 i_1 j_1 + 108}{240},$$

and the second integral has the same form. Hence, the analytic expression for the element mass matrix in this case is

$$M^e_{\alpha\mathbf{i}\beta\mathbf{j}} = \delta_{\alpha\beta} \frac{\rho \Delta x^2}{\Delta t} \frac{167 i_1^2 j_1^2 - 134(i_1^2 + j_1^2) + 5 i_1 j_1 + 108}{240} \frac{167 i_2^2 j_2^2 - 134(i_2^2 + j_2^2) + 5 i_2 j_2 + 108}{240}.$$

The 3D version is directly analogous.

2.3 Element Boundary Matrix

The boundaries require a slight modification of the above framework in both 2D and 3D.

2.3.1 2D

The boundary $\partial\Omega_D$ consists of line segments. Let $\partial\Omega^e_D$ denote such an element. Consider the mapping

$$\boldsymbol{\phi}_{Be}(\xi) = \frac{\mathbf{x}^e_0 + \mathbf{x}^e_1}{2} + \xi \left(\mathbf{x}^e_1 - \mathbf{x}^e_0 \right)$$

from $[-1/2, 1/2]$ to $\partial\Omega^e_D$, where $\mathbf{x}^e_0$ is the left (or bottom) grid node and $\mathbf{x}^e_1$ is the right (or top) grid node.

Consider a horizontal line segment. On this segment, there are only 2 grid node indices $\mathbf{b}$ and 6 cell center indices $\mathbf{i}$ for which the corresponding functions $\chi_{\mathbf{b}}$ and $N_{\mathbf{i}}$ are nonzero. So as in the case of the volume integrals, we only consider the local indices.

Let b and $\tilde{\mathbf{i}} = (i, j)$ denote local indices corresponding to $\mathbf{b}$ and $\mathbf{i}$ with $b \in \{0, 1\}$, $i \in \{-1, 0, 1\}$, and $j \in \{0, 1\}$. Here, $j = 0$ denotes cell centers below the line segment, and $j = 1$ denotes cell centers above.

Using the definitions above, we have

$$\chi_{\mathbf{b}}(\mathbf{x}) = \tilde{\chi}_b\left(\boldsymbol{\phi}_{Be}^{-1}(\mathbf{x})\right) = \tilde{\chi}_b(\xi)$$

and

$$N_{\mathbf{i}}(\mathbf{x}) = \tilde{N}_i\left(\phi_{Be1}^{-1}(\mathbf{x})\right)\tilde{N}_j\left(\phi_{Be2}^{-1}(\mathbf{x})\right) = \frac{1}{2}\tilde{N}_i\left(\xi\right).$$

Note that the last function is independent of j.

With this change of variable, we have

$$\begin{aligned}
B^e_{\mathbf{b}\alpha\mathbf{i}} &= \int_{\partial\Omega^e_D} n_\alpha \chi_{\mathbf{b}} N_{\mathbf{i}} ds(\mathbf{x}) \\
&= \int_{-1/2}^{-1/2} \tilde{\chi}_b(\xi)\frac{1}{2}\tilde{N}_i\left(\xi\right) n_\alpha ||\mathbf{x}^e_0 - \mathbf{x}^e_1|| d\xi \\
&= n_\alpha \Delta x \int_{-1/2}^{-1/2} \frac{1}{2}\tilde{\chi}_b(\xi)\tilde{N}_i\left(\xi\right) d\xi,
\end{aligned}$$

The integral has the analytic solution

$$\int_{-1/2}^{-1/2} \frac{1}{2}\tilde{\chi}_b(\xi)\tilde{N}_i\left(\xi\right) d\xi = \frac{(-6)i^2 + i(2b-1) + 8}{48},$$

so

$$B^e_{\mathbf{b}\alpha\mathbf{i}} = n_\alpha \Delta x \frac{(-6)i^2 + i(2b-1) + 8}{48}.$$

For a vertical line segment, the formula is the same except with j in place of i (note that the ranges of the indices i and j are also swapped in this case).

2.3.2 3D

For 3D, $\partial\Omega^e_D$ is a square instead of a line segment. There are three cases, depending on whether $\mathbf{n}$ in the x, y, or z direction. In any case, consider the mapping

$$\boldsymbol{\phi}_{Be}(\boldsymbol{\xi}) = \frac{\mathbf{x}^e_0 + \mathbf{x}^e_1 + \mathbf{x}^e_2 + \mathbf{x}^e_3}{4} + \xi_1(\mathbf{x}^e_2 - \mathbf{x}^e_0) + \xi_2(\mathbf{x}^e_1 - \mathbf{x}^e_0)$$

where the points $\mathbf{x}^e$ are the grid nodes incident to the square, with $\mathbf{x}^e_0$ and $\mathbf{x}^e_3$ being opposite vertices. For example, consider the case where $\mathbf{n}$ points in the x direction. Then we have $\mathbf{x}^e_1 = \mathbf{x}^e_0 + \Delta x\mathbf{e}_z$, $\mathbf{x}^e_2 = \mathbf{x}^e_0 + \Delta x\mathbf{e}_y$, and $\mathbf{x}^e_3 = \mathbf{x}^e_0 + \Delta x\mathbf{e}_y + \Delta x\mathbf{e}_z$.

Here, there are only 4 grid node indices $\mathbf{b}$ and 18 cell center indices $\mathbf{i}$ for which the corresponding functions $\chi_{\mathbf{b}}$ and $N_{\mathbf{i}}$ are nonzero.

Let $\tilde{\mathbf{b}} = (b, c)$ and $\tilde{\mathbf{i}} = (i, j, k)$ with $b, c \in \{0, 1\}$, $j, k \in \{-1, 0, 1\}$, and $i \in \{0, 1\}$ denote the corresponding local indices. Here, $i = 0$ corresponds to the 9 cell centers behind the square, and $i = 1$ corresponds to the 9 cell centers in front of the square.

Analogous to the 2D case, the formula is independent of the index i:

$$\begin{aligned}
B^e_{\mathbf{b}\alpha\mathbf{i}} &= \int_{\partial\Omega^e_D} n_\alpha \chi_{\mathbf{b}} N_{\mathbf{i}} ds(\mathbf{x}) \\
&= \int_{[-1/2,1/2]^2} \tilde{\chi}_b(\boldsymbol{\xi}) \tilde{N}_{\tilde{\mathbf{i}}}(\boldsymbol{\xi})\, n_\alpha ||(\mathbf{x}^e_2 - \mathbf{x}^e_0) \times (\mathbf{x}^e_1 - \mathbf{x}^e_0)|| d\boldsymbol{\xi} \\
&= n_\alpha \Delta x^2 \int_{-1/2}^{-1/2} \int_{-1/2}^{-1/2} \tilde{\chi}_b(\xi_1)\tilde{\chi}_c(\xi_2) \frac{1}{2} \tilde{N}_j(\xi_1)\, \tilde{N}_k(\xi_2)\, d\xi_1 d\xi_2 \\
&= n_\alpha \Delta x^2 \frac{1}{2} \left(\int_{-1/2}^{-1/2} \tilde{\chi}_b(\xi_1) \tilde{N}_j(\xi_1)\, d\xi_1 \right) \left(\int_{-1/2}^{-1/2} \tilde{\chi}_c(\xi_2) \tilde{N}_k(\xi_2)\, d\xi_2 \right).
\end{aligned}$$

The integrals are the same as the one calculated above, so we obtain the formula

$$B^e_{\mathbf{b}\alpha\mathbf{i}} = n_\alpha \Delta x^2 \frac{1}{2} \frac{(-6)j^2 + j(2b-1) + 8}{24} \frac{(-6)k^2 + k(2c-1) + 8}{24}.$$

For squares with the normal pointing in the y or z directions, the formula is obtained by making appropriately replacing j and k in this formula with the correct indices.

3 Cut Cell

Each grid node is either inside (denoted with a $-$) or outside (denoted with a $+$) according to a given level set. Between each pair of adjacent grid nodes with opposite signs, there will be a level set crossing.

Let u_1 and u_2 denote the values at two such nodes. We may approximate the location of the crossing using linear interpolation and solving for the location of the zero. Since we map the element to the square $[-1/2, 1/2]^2$, we have:

$$t = \frac{1}{2} \frac{u_1 + u_2}{u_1 - u_2}, \qquad t \in [-1/2, 1/2].$$

This gives either the x or y coordinate of the crossing (depending on whether it is a vertical or horizontal edge), and u_1 always corresponds to the left or bottom node.

3.1 Cases

In 2D, a given cell has $2^4 = 16$ possible combinations of $+$ and $-$ on the 4 grid nodes incident to that cell. We take the domain to be the triangulation produced by the Marching Squares algorithm. In 3D, there are 8 grid nodes incident to a cell and therefore $2^8 = 256$ possible combinations of $+$ and $-$, and as in 2D we use the Marching Cubes algorithm to generate the domain as a collection of tetrahedra.

3.2 2D

3.2.1 Element Divergence Matrix

Given the triangulation of the domain over the element, we compute

$$D^e_{\mathbf{d}\alpha\mathbf{i}} = \int_{\Omega^e} \chi_{\mathbf{d}} \frac{\partial N_{\mathbf{i}}}{\partial x_\alpha} d\mathbf{x} = \Delta x \int_{[-1/2,1/2]^2} \tilde{\chi}_{\tilde{\mathbf{d}}} \frac{\partial \tilde{N}_{\tilde{\mathbf{i}}}}{\partial \eta_\alpha} d\boldsymbol{\eta}$$

as a sum of integrals over the triangles.

Let K be such a triangle with vertices $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$. Consider the following mapping from the unit square to K:

$$\mathbf{r}(u,v) = \mathbf{r}_1(1-u) + [\mathbf{r}_2(1-v) + \mathbf{r}_3 v]u,$$

which is one to one on the interior of the unit square. The Jacobian determinant of this mapping is $J = 2|K|u$, so for a given function f we have the change of variable:

$$\int_K f(\boldsymbol{\eta}) d\boldsymbol{\eta} = 2|K| \int_0^1 \int_0^1 f(\mathbf{r}(u,v)) u du dv.$$

Let f equal $\tilde{\chi}_{\tilde{\mathbf{d}}} \partial \tilde{N}_{\tilde{\mathbf{i}}} / \partial \eta_\alpha$, and let $\eta_1(u,v)$ and $\eta_2(u,v)$ denote the components of $\mathbf{r}$. Note that for fixed v, both η_1 and η_2 are linear in u. Since $\tilde{\chi}_{\tilde{\mathbf{d}}}$ is a product of linear functions in η_1 and η_2, the composition $\tilde{\chi}_{\tilde{\mathbf{d}}} \circ \mathbf{r}$ is quadratic in u. Likewise, since $\tilde{N}_{\tilde{\mathbf{i}}}$ is a product of quadratics but we are taking one derivative, $\partial \tilde{N}_{\tilde{\mathbf{i}}} / \partial \eta_\alpha \circ \mathbf{r}$ is a polynomial of degree 3 in u. The integrand $f(\mathbf{r}(u,v))u$ is therefore a polynomial of degree 6 in u.

We may therefore evaluate the first iterated integral exactly using 4 point Gaussian quadrature. Let $g_K(u,v) = f(\mathbf{r}(u,v))u$ for convenience. Then

$$2|K| \int_0^1 g_K(u,v) du = 2|K| \frac{1}{2} \sum_{r=1}^{4} w_r g_K\left(\frac{x_r+1}{2}, v\right),$$

where w_r and x_r are the quadrature weights and nodes. Note that this is the form Gaussian quadrature takes on the interval $[0,1]$.

This is now a polynomial of degree 6 in v, so we may use Gaussian quadrature again to evaluate the second iterated integral:

$$2|K| \int_0^1 \int_0^1 g_K(u,v) du dv = 2|K| \frac{1}{4} \sum_{r,s=1}^{4} w_r w_s g_K\left(\frac{x_r+1}{2}, \frac{x_s+1}{2}\right).$$

Hence, the element divergence matrix is

$$D^e_{\mathbf{d}\alpha\mathbf{i}} = \frac{\Delta x}{2}\sum_K |K| \sum_{r,s=1}^{4} w_r w_s g_K\left(\frac{x_r+1}{2}, \frac{x_s+1}{2}\right).$$

3.2.2 Element Mass Matrix

On the same triangulated domain Ω^e, we compute

$$M^e_{\alpha\mathbf{i}\beta\mathbf{j}} = \delta_{\alpha\beta}\int_{\Omega^e}\frac{\rho}{\Delta t}N_{\mathbf{i}}N_{\mathbf{j}}d\mathbf{x} = \delta_{\alpha\beta}\frac{\Delta x^2}{\Delta t}\int_{[-1/2,1/2]^2}\rho\tilde{N}_{\tilde{\mathbf{i}}}\tilde{N}_{\tilde{\mathbf{j}}}d\boldsymbol{\eta}$$

as a sum of integrals over the triangles as with the element divergence matrices. The only change is that here, f equals $\rho\tilde{N}_{\tilde{\mathbf{i}}}\tilde{N}_{\tilde{\mathbf{j}}}$ and hence the integrand $f(\mathbf{r}(u,v))u$ is now a polynomial of degree 9 in u (assuming once again that ρ is constant). Increasing the number of Gaussian quadrature points to 5 allows for exact integration.

3.2.3 Element Boundary Matrix

The boundary segments will be the lines joining the points where the level set crosses the element. If the level set isocontour intersects a voxelized boundary, there will also be a boundary segment parallel to the voxelized boundary being intersected.

Let $\partial\Omega^e_D$ denote any such line segment. Since the line segment generally does not align with the grid, the local index $\tilde{\mathbf{b}}$ will range over 4 grid node indices instead of 2, and the local index $\tilde{\mathbf{i}}$ will range over 9 cell center indices instead of 6. In other words, the indexing scheme is the same as that of the 2D divergence and measure elements. We then compute

$$B^e_{\mathbf{b}\alpha\mathbf{i}} = \int_{\partial\Omega^e_D} n_\alpha\chi_{\mathbf{b}}N_{\mathbf{i}}ds(\mathbf{x}) = n_\alpha\int_{\partial\Omega^e_D}\tilde{\chi}_{\tilde{\mathbf{b}}}\left(\boldsymbol{\phi}^{-1}_{Ve}(\mathbf{x})\right)\tilde{N}_{\tilde{\mathbf{i}}}\left(\boldsymbol{\phi}^{-1}_{Ve}(\mathbf{x})\right)ds(\mathbf{x})$$

via Gaussian quadrature. Note that we have used $\boldsymbol{\phi}_{Ve}$ instead of $\boldsymbol{\phi}_{Be}$ here.

Let L be such a line segment with endpoints $\mathbf{r}_1$ and $\mathbf{r}_2$. With the usual parametrization (denoted $\mathbf{r}$) of this line segment, we have (for a given function f):

$$n_\alpha\int_L f(\mathbf{x})ds(\mathbf{x}) = |L|n_\alpha\int_0^1 f(\mathbf{r}(u))du.$$

Proceeding as before, we note that each component of $\mathbf{r}$ is a linear function of u. So $g_L(u) = \tilde{\chi}_{\tilde{\mathbf{b}}}\left(\boldsymbol{\phi}^{-1}_{Ve}(\mathbf{r}(u))\right)\tilde{N}_{\tilde{\mathbf{i}}}\left(\boldsymbol{\phi}^{-1}_{Ve}(\mathbf{r}(u))\right)$ is a polynomial of degree 6 and we use 4 point Gaussian quadrature:

$$|L|n_\alpha\int_0^1 g_L(u)du = |L|n_\alpha\frac{1}{2}\sum_{r=0}^{4} w_r g_L\left(\frac{x_r+1}{2}\right).$$

Then the element boundary matrix is

$$B^e_{\mathbf{b}\alpha\mathbf{i}} = \frac{1}{2}\sum_L |L| n_\alpha \sum_{r=0}^{4} w_r g_L\left(\frac{x_r+1}{2}\right).$$

3.3 3D

3.3.1 Element Divergence Matrix

In 3D, we compute

$$D^e_{\mathbf{d}\alpha\mathbf{i}} = \int_{\Omega^e} \chi_{\mathbf{d}} \frac{\partial N_{\mathbf{i}}}{\partial x_\alpha} d\mathbf{x} = \Delta x^2 \int_{[-1/2,1/2]^3} \tilde{\chi}_{\tilde{\mathbf{d}}} \frac{\partial \tilde{N}_{\tilde{\mathbf{i}}}}{\partial \eta_\alpha} d\boldsymbol{\eta}$$

as a sum of integrals over tetrahedra. Let K be such a tetrahedron with vertices $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, and $\mathbf{r}_4$. Consider the following mapping from the unit cube to K:

$$\mathbf{r}(u,v,w) = \mathbf{r}_1(1-u) + \left[\mathbf{r}_2(1-v) + \left[\mathbf{r}_3(1-w) + \mathbf{r}_4 w\right] v\right] u$$

which is $1-1$ on the interior of the unit cube. The Jacobian determinant of this mapping is $J = 6|K|u^2 v$, so for some function f we have the change of variable:

$$\int_K f(\boldsymbol{\eta}) d\boldsymbol{\eta} = 6|K| \int_0^1 \int_0^1 \int_0^1 f(\mathbf{r}(u,v,w)) u^2 v \, du dv dw.$$

As in the 2D case, let f equal $\tilde{\chi}_{\tilde{\mathbf{d}}} \partial \tilde{N}_{\tilde{\mathbf{i}}}/\partial \eta_\alpha$. Counting degrees as in the 2D case but noting that $\tilde{\chi}_{\tilde{\mathbf{d}}}$ and $\tilde{N}_{\tilde{\mathbf{i}}}$ are now products of 3 functions, we see that $f(\mathbf{r}(u,v,w))u^2 v$ is now a polynomial of degree 11 in u, and hence we use 6 point Gaussian quadrature. Let $g_K(u,v) = \tilde{\chi}_{\tilde{\mathbf{d}}}(\mathbf{r}(u,v,w)) \partial \tilde{N}_{\tilde{\mathbf{i}}}/\partial \eta_\alpha(\mathbf{r}(u,v,w)) u^2 v$. Then we proceed as in the 2D case and do Gaussian quadrature 3 times:

$$6|K| \int_0^1 \int_0^1 \int_0^1 g_K(u,v,w) du dv dw = 6|K| \frac{1}{8} \sum_{r,s,t=1}^{6} w_r w_s w_t g_K\left(\frac{x_r+1}{2}, \frac{x_s+1}{2}, \frac{x_t+1}{2}\right).$$

Hence, the element divergence matrix is

$$D^e_{\mathbf{d}\alpha\mathbf{i}} = \frac{3\Delta x^2}{4} \sum_K |K| \sum_{r,s,t=1}^{6} w_r w_s w_t g_K\left(\frac{x_r+1}{2}, \frac{x_s+1}{2}, \frac{x_t+1}{2}\right).$$

3.3.2 Element Mass Matrix

In this case, we once again compute

$$M^e_{\alpha\mathbf{i}\beta\mathbf{j}} = \delta_{\alpha\beta}\int_{\Omega^e}\frac{\rho}{\Delta t}N_{\mathbf{i}}N_{\mathbf{j}}d\mathbf{x} = \delta_{\alpha\beta}\frac{\Delta x^3}{\Delta t}\int_{[-1/2,1/2]^3}\rho\tilde{N}_{\tilde{\mathbf{i}}}\tilde{N}_{\tilde{\mathbf{j}}}d\boldsymbol{\eta}$$

as a sum of integrals over tetrahedra. The integrand $f(\mathbf{r}(u,v,w))u^2v$ is now a polynomial of degree 14 in u (once again assuming that ρ is constant), and we therefore need 8 point Gaussian quadrature for exact integration.

3.3.3 Element Boundary Matrix

The boundary now consists of a collection of triangular faces. If the level set isocontour intersects a voxelized boundary, the part of that boundary which is inside the level set can also be decomposed into a union of triangles. Hence, we only consider integrals over triangles here.

Let $\partial\Omega^e_D$ denote any such triangle. As with the 2D cut cell case, the boundary normal generally is not grid aligned, and hence the local index $\tilde{\mathbf{b}}$ will range over 8 grid node indices instead of 4 and the local index $\tilde{\mathbf{i}}$ will range over 27 cell center indices instead of 18. We then compute

$$B^e_{\mathbf{b}\alpha\mathbf{i}} = \int_{\partial\Omega^e_D} n_\alpha\chi_{\mathbf{b}}N_{\mathbf{i}}ds(\mathbf{x}) = n_\alpha\int_{\partial\Omega^e_D}\tilde{\chi}_{\tilde{\mathbf{b}}}\left(\boldsymbol{\phi}^{-1}_{Ve}(\mathbf{x})\right)\tilde{N}_{\tilde{\mathbf{i}}}\left(\boldsymbol{\phi}^{-1}_{Ve}(\mathbf{x})\right)ds(\mathbf{x})$$

via Gaussian quadrature. As with 2D cut cell, we have used $\boldsymbol{\phi}_{Ve}$ instead of $\boldsymbol{\phi}_{Be}$.

Let K denote such a triangle with vertices $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$. Using the same parametrization as in the 2D divergence calculation, we have (for a given function f):

$$n_\alpha\int_K f(\mathbf{x})ds(\mathbf{x}) = 2|K|n_\alpha\int_0^1\int_0^1 f(\mathbf{r}(u,v))ududv.$$

Proceeding as before, we note that each component of $\mathbf{r}$ is a linear function of u with v fixed, and a linear function of v with u fixed. So $g_K(u,v) = \tilde{\chi}_{\tilde{\mathbf{b}}}\left(\boldsymbol{\phi}^{-1}_{Ve}(\mathbf{r}(u,v))\right)\tilde{N}_{\tilde{\mathbf{i}}}\left(\boldsymbol{\phi}^{-1}_{Ve}(\mathbf{r}(u,v))\right)u$ is a polynomial of degree 10 in u, so 6 point Gaussian quadrature is sufficient to integrate exactly:

$$2|K|n_\alpha\int_0^1\int_0^1 g_K(u,v)dudv = 2|K|n_\alpha\frac{1}{4}\sum_{r,s=1}^{6}w_rw_sg_K\left(\frac{x_r+1}{2},\frac{x_s+1}{2}\right).$$

Thus, the element boundary matrix is

$$B^e_{\mathbf{b}\alpha\mathbf{i}} = \frac{n_\alpha}{2}\sum_K|K|\sum_{r,s=1}^{6}w_rw_sg_K\left(\frac{x_r+1}{2},\frac{x_s+1}{2}\right).$$

4 Standing Pool

We show here that standing pool is a solution of the system. We consider the y direction to be the vertical direction for both 2D and 3D, and assume that ρ is constant.

Let $p_{\mathbf{c}} = \rho g\left(h_0 - y_{\mathbf{c}}\right)$, where $y_{\mathbf{c}}$ is the y coordinate of grid node $\mathbf{c}$ and h_0 is the elevation of the surface, and g is the magnitude of gravity. We also set $\lambda_{\mathbf{b}} = \rho g\left(h_0 - y_{\mathbf{b}}\right)$, as λ corresponds to the pressure on the boundary $\partial\Omega_D$.

Since linear B-splines reproduce linear functions, we observe that with this choice of $p_{\mathbf{c}}$,

$$p_{\mathbf{c}}\chi_{\mathbf{c}}(\mathbf{x}) = \rho g\left(h_0 - y\right).$$

Similarly,

$$\lambda_{\mathbf{b}}\chi_{\mathbf{b}}(\mathbf{x}) = \rho g\left(h_0 - y\right).$$

Then,

$$\begin{aligned}
(\mathbf{D}^T\mathbf{P} - \mathbf{B}^T\mathbf{\Lambda})_{\alpha\mathbf{i}} &= \int_\Omega p_{\mathbf{c}}\chi_{\mathbf{c}}\frac{\partial N_{\mathbf{i}}}{\partial x_\alpha}d\mathbf{x} - \int_{\partial\Omega_D^e} n_\alpha\lambda_{\mathbf{b}}\chi_{\mathbf{b}}N_{\mathbf{i}}ds(\mathbf{x}) \\
&= \int_\Omega \rho g\left(h_0 - y\right)\frac{\partial N_{\mathbf{i}}}{\partial x_\alpha}d\mathbf{x} - \int_{\partial\Omega_D^e} n_\alpha\rho g\left(h_0 - y\right)N_{\mathbf{i}}ds(\mathbf{x}) \\
&= \int_\Omega \rho g\left(h_0 - y\right)\frac{\partial N_{\mathbf{i}}}{\partial x_\alpha}d\mathbf{x} - \int_{\partial\Omega^e} n_\alpha\rho g\left(h_0 - y\right)N_{\mathbf{i}}ds(\mathbf{x}),
\end{aligned}$$

where the last line follows from the fact that $n_\alpha\rho g\left(h_0 - y\right)$ is zero on $\partial\Omega_N^e$, the part of the boundary corresponding to $y = h_0$. Continuing,

$$\begin{aligned}
(\mathbf{D}^T\mathbf{P} - \mathbf{B}^T\mathbf{\Lambda})_{\alpha\mathbf{i}} &= \int_\Omega \rho g\left(h_0 - y\right)\frac{\partial N_{\mathbf{i}}}{\partial x_\alpha}d\mathbf{x} - \int_{\partial\Omega^e} n_\alpha\rho g\left(h_0 - y\right)N_{\mathbf{i}}ds(\mathbf{x}) \\
&= -\int_\Omega \frac{\partial}{\partial x_\alpha}\left(\rho g\left(h_0 - y\right)\right)N_{\mathbf{i}}d\mathbf{x} \\
&= -\int_\Omega \rho g_\alpha N_{\mathbf{i}}d\mathbf{x} \\
&= -\hat{g}_{\alpha\mathbf{i}},
\end{aligned}$$

for all $\alpha\mathbf{i}$, with $\mathbf{g} = [0, -g, 0]^T$. Hence,

$$\mathbf{D}^T\mathbf{P} - \mathbf{B}^T\mathbf{\Lambda} + \hat{\mathbf{g}} = \mathbf{0}.$$

If $\mathbf{W} = \mathbf{0}$, it then follows that $\mathbf{U} = \mathbf{W} = \mathbf{0}$. By the nature of the physical system, we expect that this is the only solution for the system. It's possible that there could be other numerical solutions, but in practice we observed standing pool in various geometries.